Journal of Solid Mechanics Vol. 1, No. 2 (2009) pp. 91-97

Effect of Non-ideal Boundary Conditions on Buckling of Rectangular Functionally Graded Plates

J. Mohammadi[1,*], M. Gheisary[2]

[1]Department of Mechanical Engineering, Islamic Azad University, Arak Branch, Arak, Iran
[2]Faculty of Engineering, Islamic Azad University, Khomein Branch, Khomein, Iran

Received 25 May 2009; accepted 27 July 2009

ABSTRACT

We have solved the governing equations for the buckling of rectangular functionally graded plates which one of its edges has small non-zero deflection and moment. For the case that the material properties obey a power law in the thickness direction, an analytical solution is obtained using the perturbation series. The applied in-plane load is assumed to be perpendicular to the edge which has non-ideal boundary conditions. Making use of the Linshtead-Poincare perturbation technique, the critical buckling loads are obtained. The results were then verified with the known data in the literature.

Keywords: Buckling; Functionally graded plates; Non-ideal boundary conditions; Sliding support; Perturbation

1 INTRODUCTION

RECENT studies on buckling behavior of functionally graded (FG) structures such as beams, plates and shells are mostly carried out using the classical boundary conditions (e.g. see Refs. [1-3]). The following edge conditions widely have been considered in many papers: (i) all edges clamped, (ii) all edges simply supported, (iii) two opposite edges clamped and the others free, (iv) two opposite edges simply supported and the others free, and (v) two opposite edges clamped and the others simply supported. These classical boundary conditions provide idealized edge support. Gorman [4] utilized the superposition method to obtain the buckling loads and free vibration frequencies for a family of elastically supported rectangular plates subjected to one-directional uniform in-plane loading. Two edges running in the direction of the in-plane loading were assumed to be free and lateral displacement was forbidden along the other two edges which were given uniform elastic rotational support. Pakdemirli and Boyaci [5, 6] developed the concept of non-ideal boundary conditions to the beam problem. They considered a simply supported Euler-Bernoulli beam with an intermediate supporter which allows small deflections. Aydogdu and Ece [7] reported the improved buckling loads and natural frequencies of non-ideal rectangular isotropic plates. They expanded the critical buckling loads and frequencies in perturbation series in powers of the small parameter associated with the non-ideal boundary conditions.

The objective of the present paper is understanding the effect of non-ideal boundary conditions on the buckling loads of rectangular plates which are made of functionally graded materials. The plate is assumed to have functionally graded properties in the thickness direction. Formulation is based on the assumption that the FG plate has a non-zero deflection and a non-zero moment on one of its edges. The FG plate is subjected to perpendicular loading on the edge associated with the sliding support. The Linshtead-Poincare perturbation technique is utilized to derive the differential equation system of rectangular FG plates. The governing equations are reduced to an ordinary

* Corresponding author.
E-mail address: javad_mec@yahoo.com.

differential equation using the Levy type solution method. Analytical critical buckling loads are obtained for variation of aspect ratio and inhomogeneity parameter.

2 MATHEMATICAL FORMULATIONS

A rectangular FG plate of constant thickness h with in-plane dimensions L_x and L_y, Young's modulus E, Poisson's ratio v and mass density ρ are considered. The Kirchhoff plate equation can be written as

$$D\nabla^4 w + N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} = 0 \tag{1}$$

The transverse displacement of an in-plane loaded FG plate in the dimensionless form is governed by the following differential equation

$$\overline{w}_{,\overline{xxxx}} + 2\overline{L}^2 \overline{w}_{,\overline{xxyy}} + \overline{L}^4 \overline{w}_{,\overline{yyyy}} = -\frac{N_x L_x^2}{D}\overline{w}_{,\overline{xx}} - \frac{N_y L_x^2 \overline{L}^2}{D}\overline{w}_{,\overline{yy}} \tag{2}$$

where

$$D = \frac{E_2^2 - E_1 E_3}{E_1(1-\nu^2)} \tag{3}$$

in which $\overline{L} = L_x / L_y$ describes the aspect ratio of the plate. The dimensionless variables are introduced by

$$\overline{x} = \frac{x}{L_x}, \qquad \overline{y} = \frac{y}{L_y}, \qquad \overline{w} = \frac{w}{W_r} \tag{4}$$

Here, x and y are the dimensional Cartesian coordinates measured along the adjacent edges of the plate, w is the dimensional transverse displacement, N_x and N_y are dimensional in-plane loads, W_r is any reference displacement, and D is the flexural rigidity of the FG plate. It is well-known that there are a number of different boundary conditions associated with a rectangular plate. In this study a non-zero deflection and moment are assumed to be on one of the edges of the plate. The boundary conditions along $\overline{x}=0$, $\overline{y}=0$ and $\overline{y}=1$ are simply supported and along $\overline{x}=1$ is non-ideal simply supported which are defined as

$$\begin{aligned} &\overline{w}(\overline{x},0) = \overline{w}(\overline{x},1) = \overline{w}(0,\overline{y}) = 0, && \overline{w}(1,\overline{y}) = \varepsilon f(\overline{y}) \\ &M_y(\overline{x},0) = M_y(\overline{x},1) = M_x(0,\overline{y}) = 0, && M_x(1,\overline{y}) = \varepsilon g(\overline{y}) \end{aligned} \tag{5}$$

where $0 < \varepsilon \le 1$ and $f(\overline{y})$ and $g(\overline{y})$ are continuous functions. The transverse displacement may be expressed as

$$\overline{w} = X(\overline{x})\sin(m\pi\overline{y}) \tag{6}$$

where m represents a positive integer and $X(\overline{x})$ is unknown continuous function to be determined. The assumed displacement satisfies the boundary conditions at $\overline{y}=0$ and $\overline{y}=1$. For boundary conditions at $\overline{x}=0$ and $\overline{x}=1$, we have

$$X(0) = X''(0) = 0 \tag{7}$$

$$X(1)\sin(m\pi\overline{y}) = \varepsilon f(\overline{y}) \tag{8}$$

$$[X''(1) - \nu(mn)^2 X(1)]\sin(m\pi\overline{y}) = \varepsilon g(\overline{y}) \tag{9}$$

Note that $X(\bar{x})$ depends on the variable $\bar{x}$ and the continuous functions $f(\bar{y})$ and $g(\bar{y})$ can be written as

$$f(\bar{y}) = a\sin(m\pi\bar{y}) \tag{10}$$
$$g(\bar{y}) = b\sin(m\pi\bar{y}) \tag{11}$$

where a and b are constants. Thus the boundary conditions (7-9) reduce to

$$X(0)=0, \qquad X(1)=\varepsilon a, \qquad X''(0)=0, \qquad X''(1)-\nu(m\pi)^2 X(1)=\varepsilon b \tag{12}$$

The FG rectangular plate is considered to be under uni-axial compression where the externally applied in-plane load is perpendicular to the edge associated with the sliding support. For this case of loading, we should set $N_y = 0$. Substituting Eq. (6) into Eq. (2) which gives the following ordinary differential equation

$$X^{(4)} - [2(\bar{L}mn)^2 - \frac{N_x L_x^2}{D}]X'' + (\bar{L}mn)^4 X = 0 \tag{13}$$

Using the Linshtead-Poincare perturbation technique [8], the displacement and buckling load are expanded in power series as

$$X = X_0 + \varepsilon X_1 + O(\varepsilon^2) \tag{14}$$
$$N_{x_{cr}} = N_{x0} + \varepsilon N_{x1} + O(\varepsilon^2) \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (13) and balancing the terms of same order and neglecting the terms higher order than (ε^2) lead to the following equations

$$X_0^{(4)} + (N_{x0}\frac{L_x^2}{D} - 2(\bar{L}mn)^2)X_0'' + (\bar{L}mn)^4 X_0 = 0 \tag{16}$$
$$X_1^{(4)} + (N_{x0}\frac{L_x^2}{D} - 2(\bar{L}mn)^2 X_1'' + (\bar{L}mn)^4 X_1 = -\frac{L_x^2}{D}N_{x1}X_0'' \tag{17}$$

The above equations are subject to the following boundary conditions

$$X_0(0) = X_0''(0) = X_0(1) = 0, \qquad X_0''(1) = 0 \tag{18}$$
$$X_1(0) = X_1''(0) = 0, \qquad X_1(1) = a, \qquad X_1''(1) = b + \nu(mn)^2 a \tag{19}$$

Solution of Eqs. (16) and (17) can be expressed as

$$X_0(\bar{x}) = A_0\cos(\lambda_1\bar{x}) + B_0\sin(\lambda_1\bar{x}) + C_0\cos(\lambda_2\bar{x}) + D_0\sin(\lambda_2\bar{x}) \tag{20}$$
$$X_1(\bar{x}) = A_1\cos(\lambda_1\bar{x}) + B_1\sin(\lambda_1\bar{x}) + C_1\cos(\lambda_2\bar{x}) + D_1\sin(\lambda_2\bar{x}) + \frac{n^3 L_x^2 N_{x1} D_0}{2\pi(n^4 - \bar{L}^4 m^4)D}\bar{x}\cos(n\pi\bar{x}) \tag{21}$$

where

$$\lambda_1 = \pm\frac{\sqrt{-2(\bar{L}mn)^2 + N_{x0}\frac{L_x^2}{D} + \sqrt{N_{x0}\frac{L_x^2}{D}}\sqrt{N_{x0}\frac{L_x^2}{D} - 4(\bar{L}mn)^2}}}{\sqrt{2}}i \tag{22}$$

$$\lambda_2 = \pm \frac{\sqrt{-2(\bar{L}mn)^2 + N_{x0}\frac{L_x^2}{D} - \sqrt{N_{x0}\frac{L_x^2}{D}}\sqrt{N_{x0}\frac{L_x^2}{D} - 4(\bar{L}mn)^2}}}{\sqrt{2}} i \tag{23}$$

Application of the boundary conditions (18) and (19) results in

$$A_0 = A_1 = B_0 = C_0 = C_1 = 0 \tag{24}$$

$$B_1 = \frac{n^2[\pi^2(n^2 + \nu m^2)a + b]}{\pi^2(n^4 - \bar{L}^4 m^4)\sin(\frac{\bar{L}^2 m^2}{n}\pi)} \tag{25}$$

$$N_{x0} = \frac{D(n^2 + \bar{L}^2 m^2)^2 \pi^2}{L_x^2 n^2}, \qquad \lambda_1 = \frac{\bar{L}^2 m^2}{n}\pi i, \qquad \lambda_2 = n\pi i \tag{26}$$

$$N_{x1} = \frac{-2[\pi^2(\bar{L}^4 m^4 + \nu m^2 n^2)a + n^2 b]\cos(n\pi)}{\pi n^3 D_0} \times \frac{D}{L_x^2} \tag{27}$$

Thus, the displacement function $X(\bar{x})$ can be written as

$$\begin{aligned} X(\bar{x}) = D_0 Sin(n\pi\bar{x}) + \varepsilon[&\frac{n^2[\pi^2(n^2 + \nu m^2)a + b]}{\pi^2(n^4 - \bar{L}^4 m^4)\sin(\frac{\bar{L}^2 m^2}{n}\pi)} \sin(\frac{\bar{L}^2 m^2}{n}\pi\bar{x}) + D_1 \sin(n\pi\bar{x}) \\ &- \frac{[\pi^2(\bar{L}^4 m^4 + \nu m^2)a + n^2 b]\cos(n\pi)}{\pi^2(n^4 - \bar{L}^4 m^4)} \times \frac{D}{L_x^2}\bar{x}\cos(n\pi\bar{x})] + O(\varepsilon^2) \end{aligned} \tag{28}$$

and the critical buckling load is as follows

$$N_{x_{cr}} = \frac{(n^2 + \bar{L}^2 m^2)^2 \pi^2}{n^2}\frac{D}{L_x^2} - 2\varepsilon\frac{[\pi^2(\bar{L}^4 m^4 + \nu m^2)a + n^2 b]\cos(n\pi)}{\pi n^3 D_0}\frac{D}{L_x^2} + O(\varepsilon^2) \tag{29}$$

where D_0 and D_1 are arbitrary constants. The analysis and the associated results are not valid when $\bar{L}m = n$ and $\bar{L}^2 m^2 / n = k$ with n and k being positive integers.

3 RESULTS AND DISCUSSIONS

Here, we analyzed analytically the buckling of rectangular FG plates with sliding support and material module varying only in the thickness direction. The critical buckling loads of rectangular FG plates under perpendicular loading are presented. Aydogdu and Ece [7] have carried out the buckling analysis of rectangular isotropic plates with non-ideal boundary conditions under the influence of mechanical loading. The results are simulated using the present formulation for the same isotropic rectangular plate with non-ideal simply supported boundary conditions. The results are compared with those reported by Aydogdu and Ece [7] and are listed in Table 1 for perpendicular loading. Throughout the numerical computation, we let $\varepsilon = 0.1$ and $a = b = D_0 = 1$.

The effects of the functionally graded material configuration can be seen in the coefficient of the flexural rigidity of FG plate, D, especially by the inhomogeneity parameter, k. For FG plates, when the inhomogeneity parameter increases the flexural rigidity decreases and consequently the critical buckling load decreases. Based on the assumptions of Aydogdu and Ece [7], the parameter D should set equal to unity when compared to rectangular isotropic plates. The trend in the results of critical buckling loads obtained from the present approach tally well with that of Aydogdu and Ece [7]. It may be observed that the results of ideal and non-ideal buckling loads are very close to each other. For the odd values of parameter n, the non-ideal buckling loads are higher than the ideal buckling loads. For even values of n, the results show different treatment. As the aspect ratio is increased, the critical buckling loads are increased for perpendicular loading. For isotropic case ($k = 0$), D reduces to the flexural rigidity of isotopic plate with negative sign and therefore the gradient of variations reverses. It is necessary to note that the critical buckling loads cannot be calculated for $n = 0$. Table 2 demonstrates that the critical buckling loads are not affected by the aspect ratio for $m = 0$. For example, an FG rectangular plate with non-ideal boundary conditions consist of aluminum and alumina is considered.

Table 1
Non-dimensional critical buckling loads of FG rectangular plate with sliding support for perpendicular loading

(m,n)	L_x=L_y	Ideal buckling load	Non-ideal buckling load
(1,1)	0.1	10.067 *D*	10.319 *D*
	0.2	10.674 *D*	10.927 *D*
	0.4	13.280 *D*	13.548 *D*
		13.280*	13.548*
	0.6	18.254 *D*	18.587 *D*
	0.8	26.545 *D*	27.054 *D*
	1	39.478 *D*	40.358 *D*
		39.478*	40.358*
	1.5	104.247 *D*	107.677 *D*
	2	246.740 *D*	257.040 *D*
	2.5	518.770 *D*	543.566 *D*
(1,2)	0.1	39.676 *D*	39.550 *D*
	0.2	40.271 *D*	40.145 *D*
	0.4	42.699 *D*	42.571 *D*
	0.6	46.904 *D*	46.768 *D*
	0.8	53.122 *D*	52.964 *D*
	1	61.685 *D*	61.481 *D*
	1.5	96.382 *D*	95.859 *D*
	2	157.913 *D*	156.531 *D*
	2.5	259.231 *D*	256.037 *D*
(1,3)	0.1	89.023 *D*	89.107 *D*
	0.2	89.617 *D*	89.701 *D*
	0.4	92.012 *D*	92.096 *D*
	0.6	96.074 *D*	96.161 *D*
	0.8	101.908 *D*	102.001 *D*
	1	109.660 *D*	109.769 *D*
	1.5	138.791 *D*	138.992 *D*
	2	185.329 *D*	185.785 *D*
	2.5	255.033 *D*	257.000 *D*
		255.033*	257.000*

*Ref. [7]

Table 2
Comparison of the non-dimensional critical buckling loads of FG rectangular plate with sliding support for perpendicular loading when (m,n) = (0,1)

L_x=L_y	Ideal buckling load	Non-ideal buckling load
0.1	9.869 *D*	9.932 *D*
0.2	9.869 *D*	9.932 *D*
0.4	9.869 *D*	9.932 *D*
0.6	9.869 *D*	9.932 *D*
0.8	9.869 *D*	9.932 *D*
1	9.869 *D*	9.932 *D*
1.5	9.869 *D*	9.932 *D*
2	9.869 *D*	9.932 *D*
2.5	9.869 *D*	9.932 *D*

The material properties of the aluminum and alumina that comprise the FG plate are given as E_m = 70 GPa, E_c = 380 GPa, and ν = 0.3. Also, the plate thickness is chosen to be 0.1 mm. Fig. 1 highlights the influences of inhomogeneity parameter and non-ideal boundary conditions on the variation of the critical buckling loads of rectangular plates.

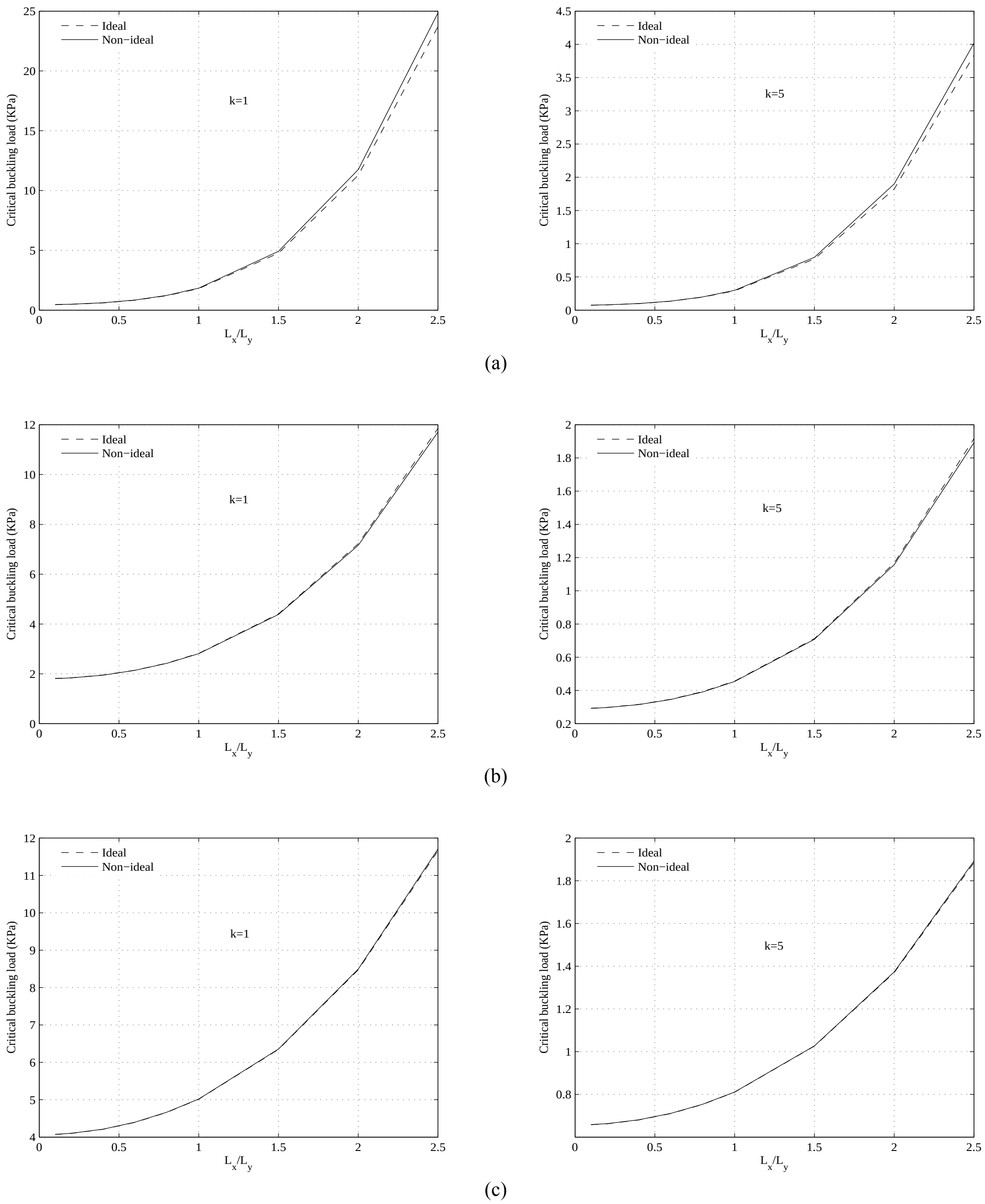

Fig. 1
Effect of non-ideal boundary conditions on the critical buckling loads of an FG rectangular plate for perpendicular loading (m = 1), (a): n = 1; (b): n = 2; (c): n = 3.

CONCLUSION

This paper presents an analytical method for buckling analysis of rectangular FG plates with non-ideal boundary conditions on one of its edges. The effects of the sliding support, aspect ratio, and inhomogeneity parameter on the critical buckling loads are discussed. The results show that the transverse displacement and critical buckling loads may increase or decrease and this is dependent on the half wave number, n. The analytical results presented in this paper serve as important benchmark values for researchers and engineers to clarify any idealized problems.

REFERENCES

[1] Reddy J.N., 2000, Analysis of functionally graded plates, *International Journal for Numerical Methods in Engineering* **47**: 663-684.
[2] Najafizadeh M.M., Eslami M.R., 2002, Buckling analysis of circular plates of functionally graded materials based on first order theory, *AIAA Journal* **40**(7): 1444-1450.
[3] Javaheri R., Eslami M.R., 2002, Thermal buckling of functionally graded plates, *AIAA Journal* **40**(1): 162-169.
[4] Gorman D.J., 2000, Free vibration and buckling of in-plane loaded plates with rotational elastic edge support, *Journal of Sound and Vibration* **229**(4): 755-773.
[5] Pakdemirli M., Boyaci H., 2002, Effect of non-ideal boundary conditions on the vibrations of continuous systems, *Journal of Sound and Vibration* **249**: 815-823.
[6] Pakdemirli M., Boyaci H., 2003, Non-linear vibrations of a simple-simple beam with a non-ideal support in between, *Journal of Sound and Vibration* **268**: 331-341.
[7] Aydogdu M., Ece M.C., 2006, Buckling and vibration of non-ideal simply supported rectangular isotropic plates, *Mechanics Research Communications* **33**: 532-540.
[8] Nayfeh A.H., 1981, *Introduction to Perturbation Techniques*, Wiley, New York.