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Static Analysis of Functionally Graded Annular Plate Resting on Elastic Foundation Subject to an Axisymmetric Transverse Load Based on the Three Dimensional Theory of Elasticity

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ABSTRACT

In this paper, static analysis of functionally graded annular plate resting on elastic foundation with various boundary conditions is carried out by using a semi-analytical approach (SSM-DQM). The differential governing equations are presented based on the three dimensional theory of elasticity. The plate is assumed isotropic at any point, while material properties to vary exponentially through the thickness direction and the Poisson's ratio remain constant. The system of governing partial differential equations can be writhen as state equations by expanding the state variables and using the state space method (SSM) about thickness direction and applying the one dimensional differential quadrature method (DQM) along the radial direction. Interactions between the plate and two parameter elastic foundations are treated as boundary conditions. The stresses and displacements distributions are obtained by solving these state equations. In this study, the influences of the material property graded index, the elastic foundation coefficients (Winkler-Pasternak), the thickness to radius ratio, and edge supports effect on the bending behavior of the FGM annular plate are investigated and discussed in details.

Keywords: FGM annular Plate; Elastic Foundation; Semi-Analytical Method; Differential Quadrature; State Space

1 INTRODUCTION

COMPONENTS made of FGMs like plates/shells resting on elastic foundations often find application in aerospace, mechanical, nuclear and offshore structures. They are, in general, subjected to various types of mechanical loads. Therefore, the modeling and analysis of mechanical behavior of plates attached to elastic foundations have been widely used by many researchers during the past decades. To describe the interactions between the plate and supporting foundation various kinds of models have proposed by scientists. The Winkler-Pasternak model is widely used to describe the mechanical behavior of structure-foundation interactions.

Functionally graded materials (FGMs) have gained considerable attention in recent years. FGMs are a new kind of composite materials and have wide applications. Since their material properties vary as a function with respect to the coordinates, their problems are more complicated than those of homogeneous materials. FGMs are composite materials that are microscopically inhomogeneous, and the mechanical properties vary continuously in one (or more) direction(s). This is achieved by gradually changing the composition of the constituent materials along one direction, usually in the thickness direction, to obtain a smooth variation of material properties and an optimum response to externally applied loading. The static and dynamic analyses of FGMs structural components are important in the design of FGMs devices. Several results can be found in the literature. For example, Reddy et al. [1] investigated the axisymmetric bending of an FGM circular plate based on the first-order plate theory and obtained the relationships

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between the first-order plate theory and the classical thin plate theory. Yang and Shen [2] dealt with the dynamic response of initially stressed functionally graded rectangular thin plates subjected to partially distribute impulsive lateral loads. Ma and Wang [3] studied the axisymmetric bending of an FGM circular plate with the third-order plate theory. Vel and Batra [4] presented a three-dimensional exact solution for free and forced vibrations of simply supported functionally graded rectangular plates. Chen [5] investigated the nonlinear vibration of functionally graded plates with arbitrary initial stresses and the effects of the amplitude of vibration, initial condition and volume fraction on nonlinear vibration were studied. Serge [6] considered the problems of free vibration, buckling, and static deflections of functionally graded plates in which material properties vary through the thickness. Park and Kim [7] investigated the thermal post buckling and vibration of the functionally graded plate considering the nonlinear temperature-dependent material properties. Li et al. [10] presented the elasticity solutions for a transversely isotropic FGM circular plate subject to an axisymmetric transverse load in terms of the polynomials of even orders. Huang, et al. [12] presented an exact solution for FG rectangular thick plates resting on elastic foundation, based on the three-dimensional theory of elasticity, using infinite dual series of trigonometric functions combined with the state- space method. Wang, Y. et al. [13] applied direct displacement method to investigate the free axisymmetric vibration of transversely isotropic Circular plate. Malekzadeh [14] used DQ method to the free vibration analysis of thick FG rectangular plates supported on two-parameter elastic foundation. Hosseini-Hashemi [15] investigated buckling and free vibration behaviors of radially functionally graded circular and annular sector thin plates subjected to uniform in-plane compressive loads and resting on the Pasternak elastic foundation by using DQ method. Nie and Zhong [8-16] investigated the bending of 2D FG circular and annular plates, the vibration of the FGM circular plates, and FGM annular sectorial plates, and dynamic behavior of 2D directional FG annular plates based on the three-dimensional theory of elasticity using the state- space method combined with the DQM.

In a survey of literature, the authors have found no work on static analysis of functionally graded annular plates resting on elastic foundation. Therefore, this paper deals with static behavior of FGMs annular plates resting on elastic foundation subject to an axisymmetric transverse load. The material properties are assumed to be graded in the thickness direction according to an exponential distribution. The formulations are based on the three-dimensional theory of elasticity. A semi-analytical method, which makes use of the state space method and the one-dimensional differential quadrature method, is employed in the static analysis. Numerical results for the static response of annular plates resting on two parameter elastic foundation are presented.

2 The MATHEMATICAL FORMULATION

Consider a transversely isotropic annular plate composed of FGMs with inner radius b and outer radius a, and thickness h, resting on two parameter elastic foundation with the Winkler stiffness of k_w and shear stiffness of k_{pi} ($i=r,\theta$), subject to an axisymmetric transverse load (uniform pressure), defined in cylindrical coordinate system r, θ, z with the origin o on the center of the bottom plane as shown in Fig. 1. The plate is assumed isotropic at any point in the volume with constant Poisson's ratio v, while the elastic stiffness components are continuous functions of the coordinates and varies exponentially along the thickness direction of the plate according to the following form,

$$c_{ij} = c_{ij}^0 \, \mathrm{e}^{\lambda\left(\frac{z}{h}\right)} \tag{1}$$

where c_{ij}^0 denotes the elastic constants of the material at the bottom plane of the plate, λ denote the mixture index along the thickness direction. Suppose that the grading of the material, applied loads, and boundary conditions are axisymmetric so that the circumferential displacement u_θ and $\partial / \partial\theta$ are identically zero.

2.1 Governing equations

The equilibrium equations of the axisymmetric problem, in the absence of body forces, are

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 , \quad \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \tag{2}$$

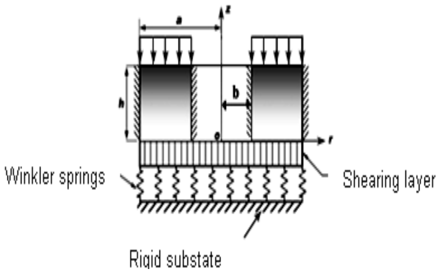

Fig. 1
Geometry of FGMs annular plate with clamped-clamped supports resting on two parameter elastic foundations.

where $\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}$ are the stress components. The strains are related to the displacements by

$$\varepsilon_{\mathrm{r}} = \frac{\partial u_r}{\partial r}, \varepsilon_\theta = \frac{u_r}{r}, \varepsilon_{\mathrm{z}} = \frac{\partial u_z}{\partial z}, \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \tag{3}$$

where u_r and u_z are the displacements components; $\varepsilon_r, \varepsilon_\theta, \varepsilon_z$ and γ_{rz} are the strain components. For a homogeneous, orthotropic material, the linear constitutive equations are

$$\begin{aligned} \sigma_r &= c_{11}\varepsilon_r + c_{12}\varepsilon_\theta + c_{13}\varepsilon_z \\ \sigma_\theta &= c_{12}\varepsilon_r + c_{11}\varepsilon_\theta + c_{13}\varepsilon_z \qquad , \ \tau_{rz} = c_{55}\gamma_{rz} \\ \sigma_z &= c_{13}\varepsilon_r + c_{13}\varepsilon_\theta + c_{33}\varepsilon_z \end{aligned} \tag{4}$$

where c_{ij} denotes the elastic constants of the material at any arbitrary plane of the plate according Eq. (1). By considering Eqs. (1)-(4) the governing differential equations at lower surface of the plate can be written in terms of displacements as follow:

$$\begin{aligned} \frac{\partial^2 u_r}{\partial z^2} &= -\frac{c_{11}^0}{c_{44}^0}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) - \frac{\lambda}{h}\frac{\partial u_z}{\partial r} - \frac{\lambda}{h}\frac{\partial u_r}{\partial z} - \frac{\partial}{\partial r}\frac{\partial u_z}{\partial z} - \frac{c_{13}^0}{c_{44}^0}\frac{\partial}{\partial r}\frac{\partial u_z}{\partial z} \\ \frac{\partial^2 u_z}{\partial z^2} &= -\frac{\lambda c_{13}^0}{h c_{33}^0}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) - \frac{c_{44}^0}{c_{33}^0}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) - \frac{c_{13}^0 + c_{44}^0}{c_{33}^0}\left(\frac{\partial}{\partial r}\frac{\partial u_r}{\partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) - \frac{\lambda}{h}\frac{\partial u_z}{\partial z} \end{aligned} \tag{5}$$

2.2 The effect of the elastic foundation

It is assumed that the bottom plane of the plate subjected to Winkler-Pasternak elastic foundation (see Fig. 1). The interactions between the plate and supporting foundation can be expressed as

$$\sigma_{z0} = k_w u_{z0} - k_p \nabla^2 u_{z0} \tag{6}$$

where σ_{z0} denotes the foundation reaction per unit area, ∇^2 is Laplace differential operator. For an axisymmetric problem, and isotropic foundation, that is, $k_{pr} = k_{p\theta} = k_p$, this model can be writhen as follow

$$\sigma_{z0} = k_w u_{z0} - k_p\left(\frac{\partial^2 u_{z0}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{z0}}{\partial r}\right) \tag{7}$$

2.3 Boundary conditions

In this study, the following cases of edges conditions are used:

(a) For an annular plate with Clamped inner and outer edges at r=b and r=a (c-c):

$$\text{at } r=b, u_r=0, u_z=0\,; \quad \text{at } r=a, u_r=0, u_z=0 \tag{8}$$

(b) For an annular plate with simply supported inner and clamped outer edges at r = b and r = a (s-c):

$$\text{at } r=b, \sigma_r=0, u_z=0\,; \quad \text{at } r=a, u_r=0, u_z=0 \tag{9}$$

(c) For an annular plate with free supported inner and clamped outer edges at r = b and r = a (f-c):

$$\text{at } r=b, \sigma_r=0, \tau_{rz}=0\,; \quad \text{at } r=a, u_r=0, u_z=0 \tag{10}$$

The boundary conditions at bottom and top surfaces of an annular plate are as follow.

at bottom surface (z=0) $\quad \tau_{rz}=0, \sigma_z=\sigma_{z0}(r)$ (11)

at top surface (z=h) $\quad \tau_{rz}=0, \sigma_z=p(r)$ (12)

where $\sigma_{z0}=(r)$ is interactions between bottom surface of the plate and foundation, $p(r)$ is external load at top surface of the plate.

3 SEMI-ANALYTICAL MEHTOD

A semi-analytical approach combining the state space method (SSM) and the one dimensional differential quadrature method (DQM) to solve the governing equations are shown in Eq. (5) The semi-analytical approach employs the state space method in the gradient direction (z-direction) of circular plates and uses the one-dimensional differential quadrature rule in the radial direction to establish a linear eigenvalue system from which the displacements and the static response can be obtained.

3.1 Formulation for the SSM

By letting the displacements u_r, u_z and their first derivatives $\partial u_r/\partial z, \partial u_z/\partial z$ be the state variables, the governing equations shown in Eq. (5) can be written as the state space formulation.

$$\frac{\partial}{\partial z}\{I\} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}\{I\} \tag{13}$$

where $I = \left\{ u_r \quad u_z \quad \partial u_r/\partial z \quad \partial u_z/\partial z \right\}^{\mathrm{T}}$, and D_1 and D_2 are given in Appendix A. By considering the following non-dimensional parameters

$$Z=\frac{z}{h}, \quad R=\frac{r}{a}, \quad U_R=\frac{u_r}{h}, \quad U_Z=\frac{u_z}{h}, \quad \bar{C}_{ij}^0=\frac{c_{ij}^0}{c_{33}^0} \tag{14}$$

The governing equations, shown in Eq. (5) can be written as

$$\frac{\partial}{\partial z}\{I\} = \begin{bmatrix} D_1 \\ D_2(R) \end{bmatrix}\{I\} \tag{15}$$

where $I=\left\{U_R \;\; U_Z \;\; \partial U_R/\partial Z \;\; \partial U_Z/\partial Z\right\}^{\mathrm{T}}$, D_1 and $D_2(R)$ are 2×4 matrices. The elements of matrix D_1 are constant and the elements of matrix $D_2(R)$ are functions of the variable *R*. In order to get the solution of Eq. (15) we need to apply the DQM approximation to the elements of matrix $D_2(R)$.

3.2 Differential quadrature method (DQM)

The differential quadrature method is a numerical solution technique for initial and/or boundary problems. The DQ method approximates the derivative of a function at any discrete point by a weighted linear summation of the functional values in the whole domain. According to this method, the *q*th derivative of a function *f*(*r*) can be approximated as [17]

$$\frac{\partial^q f(r)}{\partial r^q}\Big|_{r=r_i}=\sum_{j=1}^{N}A_{ij}^{(q)}f(r_j), \qquad i=1,2,3,...,N, \quad q=1,2,3,...,N-1 \tag{16}$$

where *N* denotes the total number of discrete points, r_i are the discrete points, $f(r_j)$ are the function values at these points. From this equation, one can conclude that the important components of DQ approximation are weighting coefficients and the choice of sampling points. The weighting coefficients for the first-order derivatives in the r-direction are determined as [17]

$$\begin{aligned} A_{ik}&=\frac{M(r_i)}{(r_i-r_k)M(r_k)} \qquad && i\neq k, \quad i,k=1,2,3,...,N \\ A_{ii}&=-\sum_{j=1,j\neq i}^{N}A_{ij} && i=k, \quad i=1,2,3,...,N \end{aligned} \tag{17}$$

where *M*(*r*) is defined as

$$M(r_i)=\prod_{j=1,j\neq i}^{N}(r_i-r_j) \tag{18}$$

The weighting coefficients of the second-order derivative can be obtained from the following relationship

$$\left[B_{ij}\right]=\left[A_{ij}\right]\left[A_{ij}\right]=\left[A_{ij}\right]^2 \tag{19}$$

In the present study, the grid points are taken non-uniformly spaced and are given by the following equation.

$$r_i=b+\frac{1}{2}\left[1-\cos\frac{(i-1)\pi}{N-1}\right](a-b), \qquad i=1,2,...,N \tag{20}$$

3.3 Application of the DQM

According to the differential quadrature rule, the partial derivatives with respect to R of the unknown functions U_R, U_Z at arbitrary point R_i can be expressed as:

$$\frac{\partial U_R}{\partial R}\Big|_{R=R_i}=\sum_{j=1}^{N}A_{ij}U_{Rj}, \qquad \frac{\partial U_Z}{\partial R}\Big|_{R=R_i}=\sum_{j=1}^{N}A_{ij}U_{Zj} \tag{21}$$

$$\frac{\partial^2 U_R}{\partial R^2}\Big|_{R=R_i}=\sum_{j=1}^{N} B_{ij}U_{Rj}, \qquad \frac{\partial^2 U_Z}{\partial R^2}\Big|_{R=R_i}=\sum_{j=1}^{N} B_{ij}U_{Zj}$$

$$\frac{\partial^2 U_R}{\partial R\partial Z}\Big|_{R=R_i}=\sum_{j=1}^{N} A_{ij}\frac{\partial U_{Rj}}{\partial Z}, \qquad \frac{\partial^2 U_Z}{\partial R\partial Z}\Big|_{R=R_i}=\sum_{j=1}^{N} A_{ij}\frac{\partial U_{Zj}}{\partial Z}$$

where $U_{Rj}, U_{Zj}, \partial U_{Rj}/\partial Z$ and $\partial U_{Zj}/\partial Z$ are the function values at the discrete point R_j, A_{ij}, B_{ij} , are the weighting coefficients of the first derivative and second derivative, respectively. By substituting Eq. (21) into (15), the following state space equation at discrete points is then obtained

$$\frac{\partial}{\partial Z}\{I_i\}=\begin{bmatrix} D_1^i \\ D_2(R_i)\end{bmatrix}\{I_i\}, \qquad i=1,2,3,...,N \tag{22}$$

where the elements of matrix $D_2(R_i)$ are constant. The elements of matrixes D_1^i and $D_2(R_i)$ are given in Appendix B. The boundary conditions shown in Eqs. (8)-(10) in discretized forms can be expressed as follows, respectively.

$$\text{at}\quad R=\frac{b}{a}, U_{R1}=0, U_{Z1}=0 \qquad\qquad \text{at}\quad R=1, U_{RN}=0, U_{ZN}=0 \tag{23}$$

$$\text{at}\quad R=\frac{b}{a}, \sigma_{R1}=0, U_{Z1}=0 \qquad\qquad \text{at}\quad R=1, U_{RN}=0, U_{ZN}=0 \tag{24}$$

$$\text{at}\quad R=\frac{b}{a}, \sigma_{R1}=0, \tau_{RZ1}=0 \qquad\qquad \text{at}\quad R=1, U_{RN}=0, U_{ZN}=0 \tag{25}$$

The boundary conditions at the top and bottom surfaces of the plate Eqs.(11) and (12) can be written in discretized forms at *Z*=0,

$$\begin{gathered}\frac{\partial U_{Ri}}{\partial Z}+\frac{h}{a}\sum_{j=1}^{N} A_{ij}U_{Zj}=0 \\ \frac{\partial U_{Zi}}{\partial Z}+\frac{h}{a}\bar{C}_{13}^{0}\left(\sum_{j=1}^{N} A_{ij}U_{Rj}+\frac{U_{Ri}}{R_i}\right)=\frac{\sigma_{Z0}(R_i)}{\bar{C}_{33}^{0}C_{33}^{0}}=K_W U_{Zi}-K_p\left(\sum_{j=1}^{N} B_{ij}U_{Zj}+\frac{1}{R_j}\sum_{j=1}^{N} A_{ij}U_{Zj}\right) \\ K_W=\frac{K_W h}{\bar{C}_{33}^{0}C_{33}^{0}}, \qquad K_P=\frac{K_P}{\bar{C}_{33}^{0}C_{33}^{0}}\frac{h}{a^2}, \qquad (i=1,2,3,...,N)\end{gathered} \tag{26}$$

where K_W and K_P are the non dimensional elastic coefficients of the foundation. At *Z*=1,

$$\begin{gathered}\frac{\partial U_{Ri}}{\partial Z}+\frac{h}{a}\sum_{j=1}^{N} A_{ij}U_{Zj}=0 \\ \frac{\partial U_{Zi}}{\partial Z}+\frac{h}{a}\bar{C}_{13}^{0}\left(\sum_{j=1}^{N} A_{ij}U_{Rj}+\frac{U_{Ri}}{R_i}\right)=\frac{-P}{e^{\lambda}\bar{C}_{33}^{0}C_{33}^{0}}, \qquad (i=1,2,3,...,N)\end{gathered} \tag{27}$$

By implement the boundary conditions in Eq. (22), the solution to Eq. (15) can be written as:

$$I_i(Z)=e^{M_i Z}I_i(0) \tag{28}$$

$$M_i = \begin{bmatrix} D_1^i \\ D_2(R_i) \end{bmatrix}_{4(N-2)\times 4(N-2)}$$

$$I_i(Z) = \left\{ U_{Ri}(Z) \quad U_{Zi}(Z) \quad \frac{\partial U_{Ri}(Z)}{\partial Z} \; \frac{\partial U_{Zi}(Z)}{\partial Z} \right\}^T \qquad i = 2,3,...,N-1$$

$$I_i(0) = \left\{ U_{Ri}(0) \quad U_{Zi}(0) \quad \frac{\partial U_{Ri}(0)}{\partial Z} \; \frac{\partial U_{Zi}(0)}{\partial Z} \right\}^T \tag{29}$$

From Eq. (28), we get

$$I_i(1) = e^{M_i} I_i(0) \tag{30}$$

where e^{M_i} is the global transfer matrix, and $I_i(1)$ are the values of the state variables at the top plane Z=1. Eq. 30 forms the global transfer relation of the state vectors at the bottom and top surfaces of the plates. Substituting Eqs. (26) and (27) and the corresponding boundary conditions shown in Eqs. (23)-(25) for different plates in to Eq. (30), the algebraic equations for static analysis can be obtained

$$G.T = Q \tag{31}$$

where G is a 4(N-2)×4(N-2) matrix, Q is a traction force vector and

$$T = \left[\bar{U}_R^{ii}(0) \quad \bar{U}_Z^{ii}(0) \quad \bar{U}_R^{ii}(1) \quad \bar{U}_Z^{ii}(1) \right]^T, \qquad (ii = 2,3,...,N-1) \tag{32}$$

By solving Eq. 31, all state parameters at Z=0, Z=1 are obtained. We can use the Eqs. (28) and (4) to calculate the displacements and the stresses in FGMs annular plates.

4 NUMERICAL RESULTS

For numerical illustration, an annular plate (a=1.0 m, b=0.1 m, h=0.04 a) with clamped-clamped support conditions, on linear elastic foundations is considered. The material properties are assumed as the exponential distributions in the thickness of the plate shown in Eq. (1). The boundary conditions on the top and the bottom surfaces of the plate are

$$\text{at } Z = 0, \tau_{rz} = 0, \sigma_z = \sigma_{z0}, \qquad \text{at } Z = h, \tau_{rz} = 0, \sigma_z = -1 \text{ GPa} \tag{33}$$

Effects of the material property graded index, the thickness to radius ratio, foundation stiffness, and the edge supports effect on bending behavior of a FG annular plate will be extensively discussed in the following text. The numerical results are shown in Figs. 2-7.

4.1 Convergence of the DQ method and validation

Since analytical and numerical solutions do not exist for the bending response of FGMs circular plates resting on elastic foundations in literature; hence, to show the effect of the number of the selected discrete points, convergence of the DQ method is conducted firstly, and is used as an evaluation criterion. The non-dimensional transverse deflection (U_{Z0}) vs. number of discrete points (N) at a point located at r/a=0.55 is depicted in Fig. (2). To achieve the numerical data in Fig. 2 $\lambda = 1$, $E_0 = 70$ GPa, $\nu = 0.3$, $K_W = K_P = 1$ and h/a=0.04 are considered. It can be seen from Fig.2.that the non-dimensional deflection of the plate at the midpoint of radius approaches to a specific value with an increase in the number of the discrete points. Non-equally spaced discrete points are adopted and the number of discrete points in the radial direction is eleven. Fig.2. confirms that the convergence of this method is great.

For validation, a comparison is done between the present solution and the available results in the literature. For this purpose, the results derived for a solid circular plate with clamped edge support and K_w=0, K_p=0, h/a=0.1, $E_0 = 380$ GPa, $\lambda = 1$. The variation of the radial stress component σ_r (GPa) versus z/h at a point located at r/a=0.5, for this plate are illustrated in Fig.3 and is compared with the results of Ref. [9]. From Fig. 3 it can be found that the present results are in good agreement with the available results in Ref. [9].

4.2 The effect of material property graded index

In this stage, effects of the material property graded index on bending behavior of the plate (clamped-clamped support) are considered. The through-thickness distributions of displacements and stresses of the plate ($E_0 = 70$ GPa, $\nu = 0.3$) on elastic foundation are plotted in Fig. 4. It is seen from Fig. 4. that displacements and stresses distribution through the thickness of the plate are nonlinear, and all displacements decrease, and all stresses increase gradually as λ increase. Decrease of displacements; indicate that increasing the gradient index will certainly enhance the deformation rigidity of the plate. The neutral surface of FGMs plate is not at the mid-surface but depends on the through-thickness of Young's moduli.

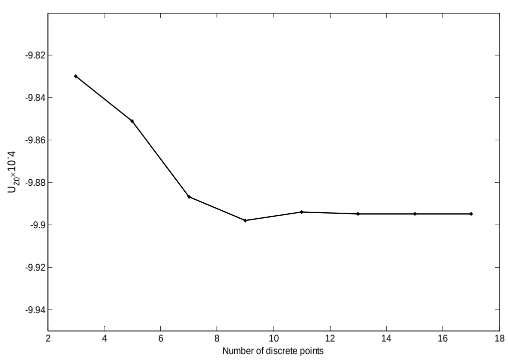

Fig. 2
Non-dimentional deflections of the plate at R=0.55 for different numbers of the discrete points

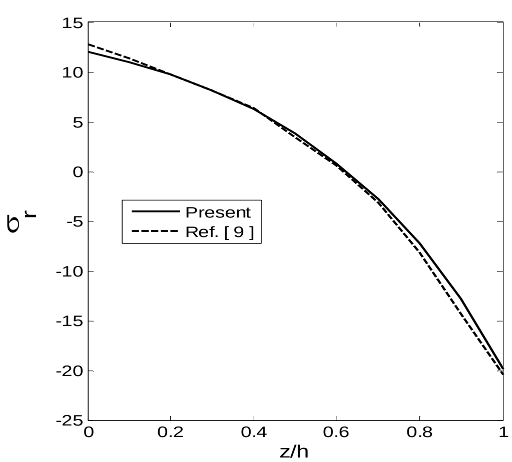

Fig. 3
Comparison of the radial stress component σ_r (GPa) versus z/h at a point located at (r/a = 0.5) for a solid circular plate (K_w=0, K_p=0) with h/a = 0.1, $E_0 = 380$ GPa, $\lambda = 1$.

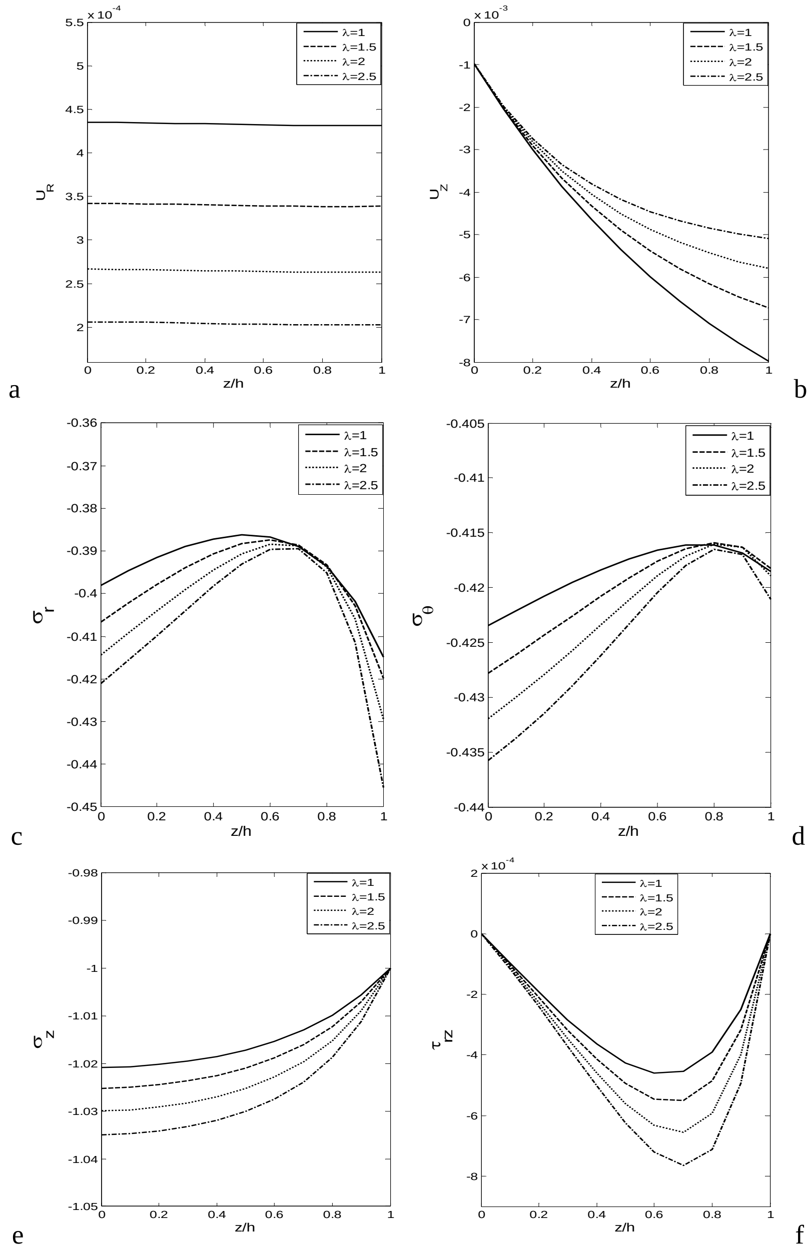

Fig. 4
Effect of the material property graded index on variation of displacement and stress components versus z/h at a point located at (r/a = 0.55) for an annular plate resting on elastic foundation (K_w=1, K_p=1) with *h*/*a*=0.04: a) radial displacement component (UR), (b) transverse displacement component (UZ), (c) radial stress component σ_r (GPa), (d) tangential stress component σ_θ (GPa), (e) transverse normal stress component σ_Z (GPa), (f) transverse shear stress component τ_{rz} (GPa).

4.3. The effect of thickness to radius ratio

The effects of the thickness to radius ratio on static behavior of the plate are plotted in Fig. 5. It is seen from Fig. 5 that U_R, U_Z, σ_r and σ_θ decrease, and stresses σ_Z, τ_{rz} increase gradually as *h*/*a* ratio increase. The distribution of transverse normal and shear stresses through the thickness of the plate converges to the horizontal lines with decreasing the thickness of the plate. It is obvious, that increasing the *h*/*a* ratio will enhance the deformation rigidity of the plate.

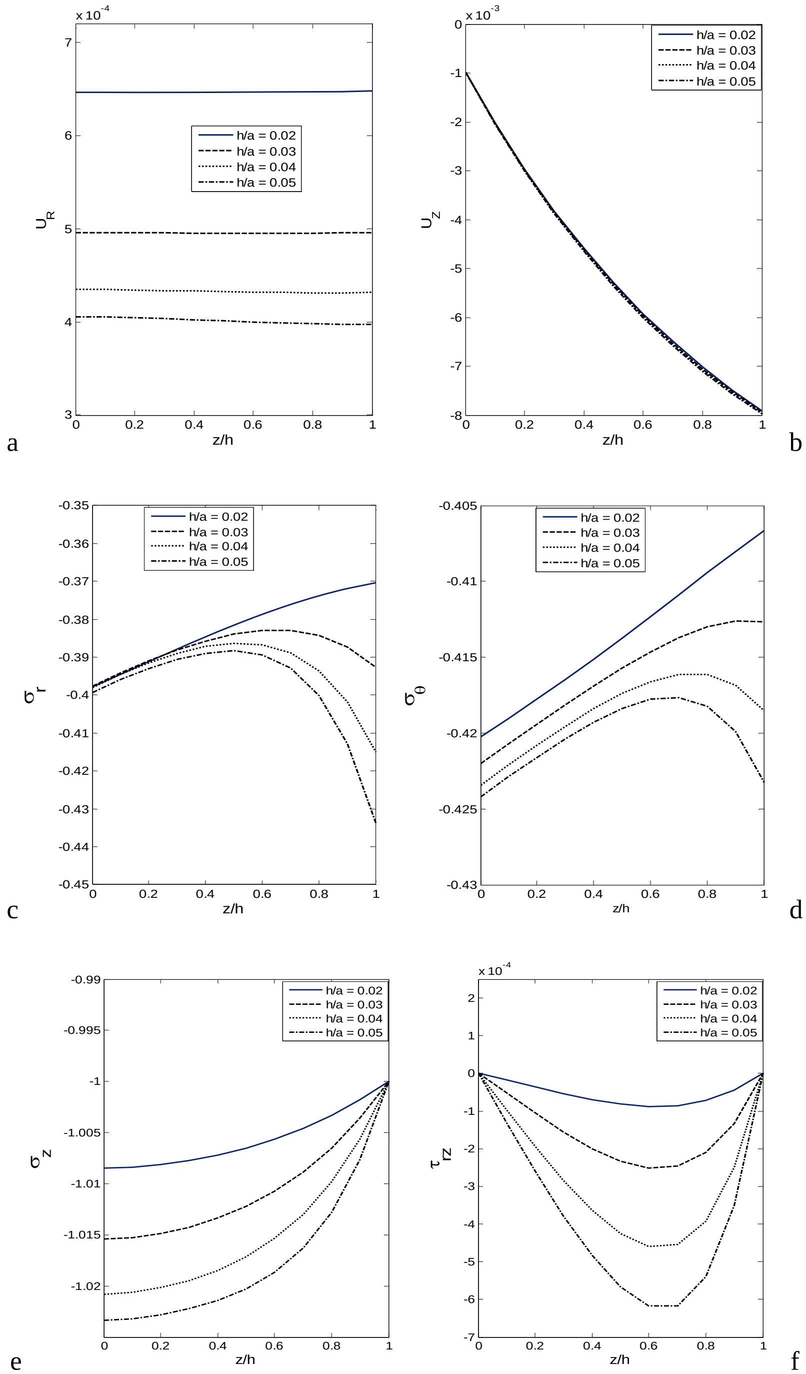

Fig. 5
Effect of the thickness per radius ratio on variation of displacement and stress components versus *z/h* at a point located at (*r*/*a*=0.55) for an annular plate resting on elastic foundation (K_w=1, K_p=1) with λ = 1: a) radial displacement component (UR), (b) transverse displacement component (UZ), (c) radial stress component σ_r (GPa), (d) tangential stress component σ_θ (GPa), (e) transverse normal stress component σ_Z (GPa), (f) transverse shear stress component τ_{rz} (GPa).

4.4. The effect of foundation stiffnesses

Effects of the foundation stiffness for the plate, and the metal rich plane attached to the elastic foundation on physical quantities are plotted in Fig. 6. It can be found from Fig. 6, that U_R, U_Z, σ_r and σ_θ decrease, and stresses σ_Z, τ_{rz} increase through the thickness direction of the circular FG plate with increasing K_w, K_p. The effect of K_w on physical quantities of the plate more than the effect of K_p on the same physical quantities.

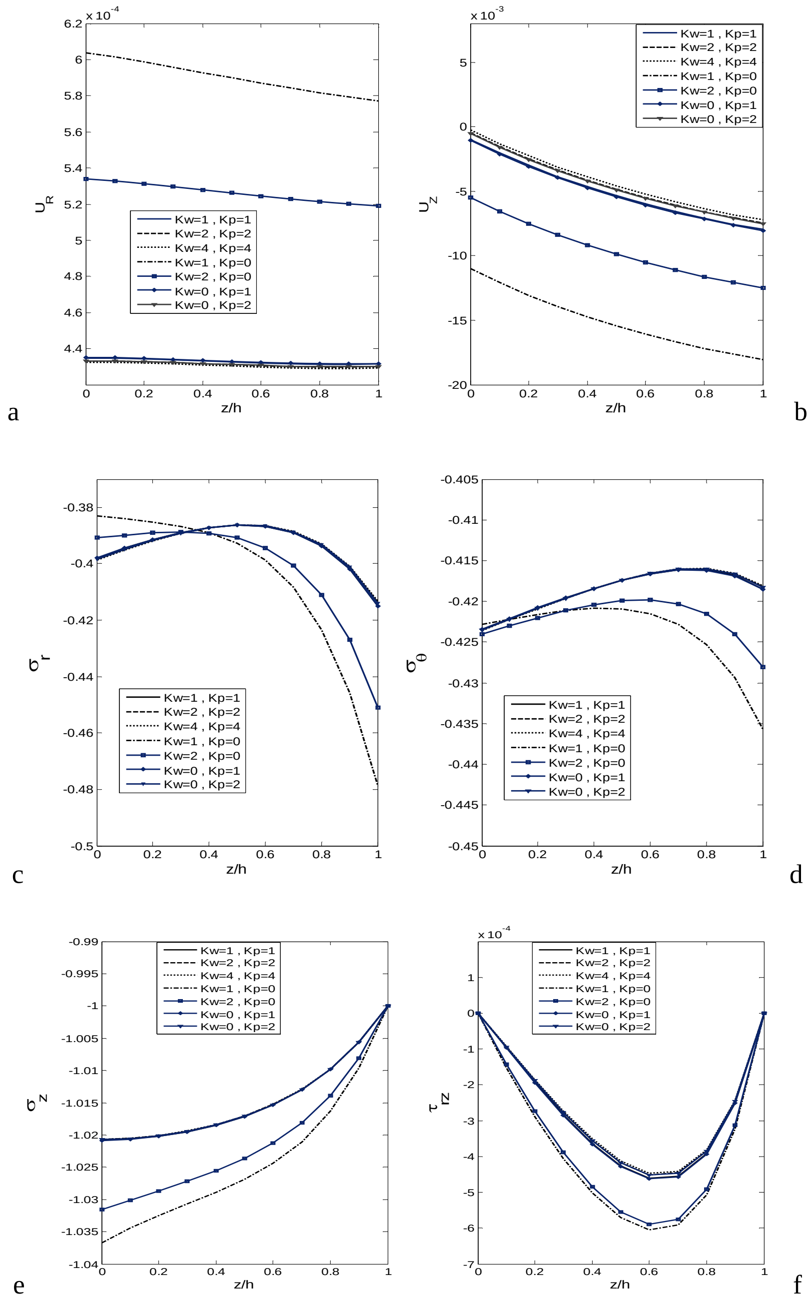

Fig. 6
Effect of the elastic foundation coefficients on variation of displacement and stress components versus *z*/*h* at a point located at (r/a = 0.55) for an annular plate resting on elastic foundation with $\lambda = 1$, h/a=0.04 : a) radial displacement component (UR), (b) transverse displacement component (UZ), (c) radial stress component σ_r (GPa), (d) tangential stress component σ_θ (GPa), (e) transverse normal stress component σ_Z (GPa), (f) transverse shear stress component τ_{rz} (GPa).

4.5. The effect of edges supports

Effect of the edges supports on variation of physical quantities through thickness for the plates $(E_0 = 70\text{ GPa},\ E_h = 380\text{ GPa})$, with different supports are plotted in Fig. 7.

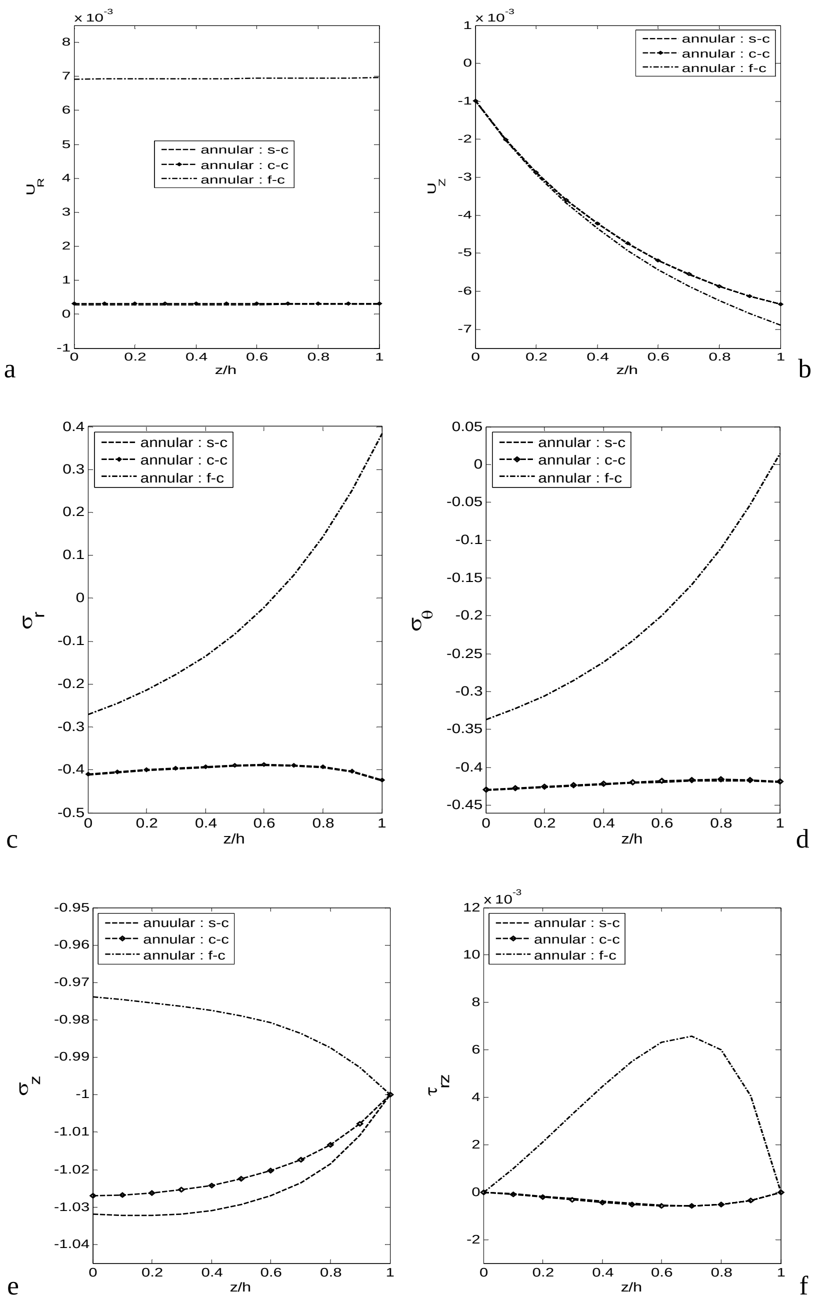

Fig. 7
Effect of supports on variation of displacement and stress components versus z/h at a point located at (r/a = 0.55) for annular plates resting on elastic foundation with $\lambda = \text{Ln}\left(E_h / E_0\right)$ h/a = 0.04, (metal rich plane on the foundation) : a) radial displacement component (UR), (b) transverse displacement component (UZ), (c) radial stress component σ_r (GPa), (d) tangential stress component σ_θ (GPa), (e) transverse normal stress component σ_Z (GPa), (f) transverse shear stress component τ_{rz} (GPa)

It is seen from Fig. 7 that maximum variation of physical quantities through thickness of the plates at a desired location occurs in free-clamped annular plates. Variation scheme of stresses through thickness for the free-clamped annular plate resting on linear elastic foundations is different with variation scheme of these quantities to other plates.

5 CONCLUSIONS

Axisymmetric bending of functionally graded annular plates resting on Winkler-Pasternak elastic foundations with various boundary conditions is investigated in this paper using semi-analytical approach (SSM-DQM). The analytical solution in the graded direction can be acquired using the state space method and approximate solution in the radial direction can be obtained using the one-dimensional differential quadrature method. By using this method some results are derived, with the most important conclusions that, (1)-the numerical results have specified that the material in homogeneity has an important effect on the bending behavior of the plate on elastic foundation. (2)-displacements and stresses varying nonlinear through thickness of the plate.(3)- the neutral surface of the FGMs plate is not at the mid-surface but depends on the through-thickness variation of Young's moduli. (4)-increasing the gradient index will certainly enhance the deformation rigidity of the plate. (5)-with decreasing the thickness of the FGMs plate, the rigidity of the plate decreases, which is the characteristic of thin plate. (6)-the effect of Kw on displacement and stress components of the plate is more than the effect of K_p on the same physical quantities. (7)-in the presence of elastic field the transverse normal stress through thickness of the plate varies gradually from specified value at bottomed surface (foundation interactions) to another value (external load) at top surface. (8)-maximum variation of physical quantities through thickness of the plates occurs to free-clamped annular plate. (9)-Variation scheme of stresses through thickness for the free-clamped annular plate is different with variation scheme of these quantities to the plate with other supports.

6 APPENDIX

Appendix A

Elements of the matrices D_1, D_2

The matrixes D_1, D_2 are given as

$$D_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \end{bmatrix} \tag{A.1}$$

where

$$d_{11} = -\frac{c_{11}^0}{c_{44}^0}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) \qquad d_{12} = -\frac{\lambda}{h}\frac{\partial}{\partial r}, \quad d_{13} = -\frac{\lambda}{h}, \quad d_{14} = -\frac{\partial}{\partial r} - \frac{c_{13}^0}{c_{44}^0}\frac{\partial}{\partial r}$$
$$d_{21} = -\frac{\lambda}{h}\frac{c_{13}^0}{c_{33}^0}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right), \quad d_{22} = -\frac{c_{44}^0}{c_{33}^0}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right), \quad d_{23} = -\frac{c_{13}^0 + c_{44}^0}{c_{33}^0}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right), \quad d_{24} = -\frac{\lambda}{h} \tag{A.2}$$

Appendix B

Elements of the matrices $D_1^i, D_2(R_i)$

$$D_1^i = \begin{bmatrix} 0 & 0 & \left[\delta_{ij}\right]_{N\times N} & 0 \\ 0 & 0 & 0 & \left[\delta_{ij}\right]_{N\times N} \end{bmatrix}_{2N\times 4N}, \quad \text{Where} \quad \delta_{ij} = 0\left(i \neq j\right), \delta_{ij} = 1$$

$$D_2\left(R_i\right) = \begin{bmatrix} \left[d_{ij}^{11}\right]_{N\times N} & \left[d_{ij}^{12}\right]_{N\times N} & \left[d_{ij}^{13}\right]_{N\times N} & \left[d_{ij}^{14}\right]_{N\times N} \\ \left[d_{ij}^{21}\right]_{N\times N} & \left[d_{ij}^{22}\right]_{N\times N} & \left[d_{ij}^{23}\right]_{N\times N} & \left[d_{ij}^{24}\right]_{N\times N} \end{bmatrix}_{2N\times 4N} \tag{B.3}$$

$$d_{ij}^{11} = -\frac{c_{11}^0}{c_{44}^0}\left(\frac{h}{a}\right)^2 \left(\sum_{j=1}^{N} B_{ij} + \frac{1}{R_i}\sum_{j=1}^{N} A_{ij} - \frac{1}{R_i^{\,2}} \right)$$

$$d_{ij}^{12} = -\frac{\lambda h}{a}\sum_{j=1}^{N} A_{ij}, \quad d_{ij}^{13} = -\lambda\ , \quad d_{ij}^{14} = -\left(\frac{h}{a}\right)\left(1 + \frac{c_{13}^0}{c_{44}^0}\right)\sum_{j=1}^{N} A_{ij}$$

$$d_{ij}^{21} = -\frac{\lambda h}{a}\frac{c_{13}^0}{c_{33}^0}\left(\sum_{j=1}^{N} A_{ij} + \frac{1}{R_i} \right), \quad d_{ij}^{22} = -\frac{c_{44}^0}{c_{33}^0}\left(\frac{h}{a}\right)^2 \left(\sum_{j=1}^{N} B_{ij} + \frac{1}{R_i}\sum_{j=1}^{N} A_{ij} \right)$$

$$d_{ij}^{23} = -\frac{c_{13}^0 + c_{44}^0}{c_{33}^0}\left(\frac{h}{a}\right)\left(\sum_{j=1}^{N} A_{ij} + \frac{1}{R_i} \right), \quad d_{ij}^{24} = -\lambda \tag{B.4}$$

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