Journal of Solid Mechanics Vol. 3, No. 1 (2011) pp. 64-73

Large Amplitude Vibration of Imperfect Shear Deformable Nano-Plates Using Non-local Theory

S.K. Jalali, A. Rastgoo*, I. Eshraghi

School of Mechanical Engineering, College of Engineering, University of Tehran, Tehran, Iran

Received 8 January 2011; accepted 2 March 2011

ABSTRACT

In this study, based on nonlocal differential constitutive relations of Eringen, the first order shear deformation theory of plates (FSDT) is reformulated for vibration of nano-plates considering the initial geometric imperfection. The dynamic analog of the von Kármán nonlinear strain-displacement relations is used to derive equations of motion for the nano-plate. When dealing with nonlinearities, in the frame work of nonlocal theory, challenges are presented because of the coupling between nonlocal stress resultants and displacement terms. Governing equations are solved using differential quadrature method (DQM) and numerical results for free vibration of an imperfect single layered graphene sheet are presented.

Keywords: Nonlocal Theory; Shear Deformable Nano Plates; Nonlinear Vibration; Geometric Imperfection.

1 INTRODUCTION

NANO structures attract more attention in recent years due to their superior mechanical, electronic, and chemical characteristics [1]. These properties have lead to widespread applications of these structures in NEMS devices. Vibration is the key phenomenon in some NEMS devices including oscillators, mass measuring sensors etc [2, 3] and overviewing the literature reveals that Carbon nanotubes (CNTs) and graphene sheets (GSs) have the most potential to be used for these devices. Thus, their vibration characteristics have been studied by many researchers [4-6]. A review concerning the importance and modeling of vibration behavior of various nanostructures has been conducted by Gibson's et al [7].

Experimental investigations using nanoscale specimen needs considerable efforts and too much expense. Therefore, various continuum models have been proposed for mathematical modeling and analysis of nano-structures. In this way, for example, a graphene sheet can be considered as a nano-plate and known theories of plates like classical (CLPT) and first order shear deformation plate (FSDT) theories can be adopted to model the displacements in the structure. Recently, Reddy [8, 9] used nonlocal differential constitutive relations of Eringen and von Kármán nonlinear strains to reformulate beam and plate theories for analysis of bending, buckling and vibration behavior of these structures. Pradhan and Phadikar [10] employed nonlocal elasticity theory for vibration analysis of single and multi layered nano-plates based on CLPT and FSDT. Navier solutions were obtained for natural frequencies with simply supported boundary conditions and effects of nonlocal parameter, length and elastic modulus were investigated. However, the effects of nonlinearity were absent in their calculations. Free in-plane vibration of nano-plates was studied using nonlocal continuum mechanics by Pradhan and Murmu [11]. Explicit statements of natural frequencies were obtained by separation of variables and the importance of including nonlocal parameter for in-plane vibration of graphene sheets was shown. Multilayered graphene sheet embedded in an elastic medium was modeled and its natural frequencies and mode shapes were obtained by Behfar and Naghdabadi [12].

* Corresponding author. Tel.: +98 912 109 6133.
E-mail address: *arastgo@ut.ac.ir* (A. Rastgoo).

Liew et al. [13] considered vibration behavior of multilayered graphene sheets embedded in an elastic matrix. They used a continuum based plate model for their analysis.

Previous researches show that initial geometric imperfection has significant effects on nonlinear vibration frequencies and shape modes of structures (see [14, 15]). Geometric imperfection refers to globally or locally distributed, small and inevitable deviations between the actual shape and intended shape of the structure. To our knowledge, the effect of geometric imperfection has not been considered in continuum formulations of nano-scale structures with consideration of non-local elasticity despite of the need for highly accurate results in nano-scale calculations.

When dealing with nonlinearities and considering imperfections, in the frame work of nonlocal theory, challenges are presented because of the coupling between various nonlocal stress resultants and displacement terms. This paper aims to derive the governing equations of motion for large amplitude nonlinear vibrations of nano plates with initial geometric imperfections using FSDT and based on nonlocal continuum elasticity. Then, these equations are solved for the case of free vibration of a clamped graphene plate while taking into account the effects of imperfection and non-local parameter.

2 THE PLATE MODEL

Considering a nano-plate of thickness h and dimension $a \times b$ (Fig. 1). Displacement field based on first order shear deformation plate theory can be written as [14]:

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\theta_x(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) + z\theta_y(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) + w^*(x,y) \end{aligned} \tag{1}$$

where u_0, v_0, w_0 represent mid-plane displacements in the x, y, and z directions respectively. θ_x, θ_y are the rotations of the normal to this plane around y and x axes respectively. w^* is the initial geometric imperfection of the plate. For nonlinear analysis von Kármán type of strain-displacement relationship is adopted here as [14]:

$$\begin{aligned} \varepsilon_{xx} &= u_{0,x} + \frac{1}{2}(w_{0,x})^2 + w_{0,x}w^*_{,x} + z\theta_{x,x} \\ \varepsilon_{yy} &= v_{0,y} + \frac{1}{2}(w_{0,y})^2 + w_{0,y}w^*_{,y} + z\theta_{y,y} \\ 2\varepsilon_{xy} &= u_{0,y} + v_{0,x} + w_{0,x}w_{0,y} + w_{0,x}w^*_{,y} + w_{0,y}w^*_{,x} + \theta_{x,y} + \theta_{y,x} \\ 2\varepsilon_{xz} &= w_{0,x} + \theta_x, \qquad 2\varepsilon_{yz} = w_{0,y} + \theta_y \end{aligned} \tag{2}$$

where $(.)_{,\alpha}$ represents differentiation with respect to α direction and a dot superscript over a variable indicates differentiation with respect to time.

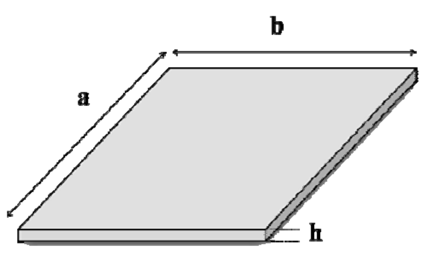

Fig. 1
Geometric representation of a continuum model of a nano-plate.

3 NONLOCAL CONTINUUM MECHANICS

3.1 Differential constitutive relations

Classical stress tensor at a reference point is a function of the configuration of the strain at that point only. However, when dealing with nonlocal stress tensor, it is related to the strain field at every point in the continuum. Considering a reference point X, the nonlocal stress tensor can be expressed as [16]:

$$\sigma^{nl} = \int_V \chi(\left| Y - X \right|, \tau)\sigma(Y)\,\mathrm{d}Y \tag{3}$$

where σ^{nl} represents nonlocal stress tensor, Y is an arbitrary point in the medium and σ stands for classical stress tensor which is related to the strain tensor through Hook's law for a linearly elastic material. It can be shown (see Eringen [16]) that the integral constitutive relation can be equivalently represented by the following form:

$$L(\sigma^{nl}) = \sigma, \qquad L = 1 - \mu\nabla^2 \tag{4}$$

where $\mu = e_0^2 a_0^2$ is called the nonlocal parameter, a_0 is an internal characteristic length and e_0 is a constant. ∇^2 is the 2D Laplacian operator which is equal to $(.)_{,xx} + (.)_{,yy}$ in the Cartesian coordinate system.

3.2 Nonlocal stress resultants

Employing Hook's law, Eq. (4) can be rewritten as:

$$\begin{aligned}
&\sigma_{xx}^{nl} - \mu\nabla^2\sigma_{xx}^{nl} = Q_{11}\varepsilon_{xx} + Q_{12}\varepsilon_{yy}\\
&\sigma_{yy}^{nl} - \mu\nabla^2\sigma_{yy}^{nl} = Q_{12}\varepsilon_{xx} + Q_{22}\varepsilon_{yy}\\
&\sigma_{xy}^{nl} - \mu\nabla^2\sigma_{xy}^{nl} = 2Q_{66}\varepsilon_{xy}\\
&\sigma_{yz}^{nl} - \mu\nabla^2\sigma_{yz}^{nl} = 2Q_{44}\varepsilon_{yz}\\
&\sigma_{xz}^{nl} - \mu\nabla^2\sigma_{xz}^{nl} = 2Q_{55}\varepsilon_{xz}
\end{aligned} \tag{5}$$

where Q_{ij} are the reduced stiffness coefficients for plane state of stress defined by [8]:

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2}, \qquad Q_{12} = \frac{\nu E}{1-\nu^2}, \qquad Q_{44} = Q_{55} = Q_{66} = G \tag{6}$$

By substituting strain-displacement relations from Eq. (2) into nonlocal Eq. (5) and integrating along plate thickness, the following relations for the plate stress resultants will be obtained:

$$\begin{Bmatrix} N_{xx}^{nl} \\ N_{yy}^{nl} \\ N_{xy}^{nl} \end{Bmatrix} - \mu\nabla^2 \begin{Bmatrix} N_{xx}^{nl} \\ N_{yy}^{nl} \\ N_{xy}^{nl} \end{Bmatrix} = \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} u_{0,x} + \frac{1}{2}w_{0,x}^2 + w_{0,x}w_{,x}^* \\ v_{0,y} + \frac{1}{2}w_{0,y}^2 + w_{0,y}w_{,y}^* \\ u_{0,y} + v_{0,x} + w_{0,x}w_{0,y} + w_{0,x}w_{,y}^* + w_{0,y}w_{,x}^* \end{Bmatrix} \tag{7}$$

$$\begin{Bmatrix} M_{xx}^{nl} \\ M_{yy}^{nl} \\ M_{xy}^{nl} \end{Bmatrix} - \mu\nabla^2 \begin{Bmatrix} M_{xx}^{nl} \\ M_{yy}^{nl} \\ M_{xy}^{nl} \end{Bmatrix} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \theta_{x,y} + \theta_{y,x} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} Q_y^{nl} \\ Q_x^{nl} \end{Bmatrix} - \mu\nabla^2 \begin{Bmatrix} Q_y^{nl} \\ Q_x^{nl} \end{Bmatrix} = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = K_s \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} w_{0,y} + \theta_y \\ w_{0,x} + \theta_x \end{Bmatrix} \tag{9}$$

where nonlocal stress and moment resultants are defined as [10]:

$$\begin{Bmatrix} N_{xx}^{nl} \\ N_{yy}^{nl} \\ N_{xy}^{nl} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx}^{nl} \\ \sigma_{yy}^{nl} \\ \sigma_{xy}^{nl} \end{Bmatrix} \mathrm{d}z, \qquad \begin{Bmatrix} M_{xx}^{nl} \\ M_{yy}^{nl} \\ M_{xy}^{nl} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx}^{nl} \\ \sigma_{yy}^{nl} \\ \sigma_{xy}^{nl} \end{Bmatrix} z\,\mathrm{d}z, \qquad \begin{Bmatrix} Q_y^{nl} \\ Q_x^{nl} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{yz}^{nl} \\ \sigma_{xz}^{nl} \end{Bmatrix} \mathrm{d}z \tag{10}$$

and the classical local stress resultants are [8]:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \mathrm{d}z, \qquad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\,\mathrm{d}z, \qquad \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \mathrm{d}z \tag{11}$$

The parameter K_s is the shear correction factor which usually is set to 5/6 [8]. A_{ij} and D_{ij} represent the extensional and bending stiffnesses of the plate and are defined below [8]:

$$(A_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z^2) Q_{ij}\,\mathrm{d}z = \left(h, \frac{h^3}{12}\right) Q_{ij} \tag{12}$$

3.3 Governing equations

By using the virtual work principle, the governing equations for motion of a plate can be written as [8]:

$$N_{xx,x}^{nl} + N_{xy,y}^{nl} + f_x = I_0 \ddot{u}_0 \tag{13}$$

$$N_{xy,x}^{nl} + N_{yy,y}^{nl} + f_y = I_0 \ddot{v}_0 \tag{14}$$

$$Q_{x,x}^{nl} + Q_{y,y}^{nl} + N^{nl} + q = I_0 \ddot{w}_0 \tag{15}$$

$$M_{xx,x}^{nl} + M_{xy,y}^{nl} - Q_x^{nl} = I_2 \ddot{\theta}_x \tag{16}$$

$$M_{xy,x}^{nl} + M_{yy,y}^{nl} - Q_y^{nl} = I_2 \ddot{\theta}_y \tag{17}$$

where

$$N^{nl} = (N_{xx}^{nl}(w_0 + w^*)_{,x} + N_{xy}^{nl}(w_0 + w^*)_{,y})_{,x} + (N_{xy}^{nl}(w_0 + w^*)_{,x} + N_{yy}^{nl}(w_0 + w^*)_{,y})_{,y} \tag{18}$$

f_x, f_y are in-plane external forces and q is the transverse distributed load. I_0 and I_2 are mass moments of inertia and are defined as:

$$(I_0, I_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z^2)\, \rho\,\mathrm{d}z = \left(h, \frac{h^3}{12}\right) \rho \tag{19}$$

In which, ρ is the density of the constituent material. In order to rewrite the governing equations in terms of displacement field, the operator L is applied to Eqs. (13-17), which yields:

$$N_{xx,x} + N_{xy,y} + L(f_x) = I_0 \; \mathrm{L}(\ddot{u}_0) \tag{20}$$

$$N_{xy,x} + N_{yy,y} + L(f_y) = I_0 \, \mathrm{L}(\ddot{v}_0) \tag{21}$$

$$Q_{x,x} + Q_{y,y} + L(N^{nl}) + \mathrm{L}(q) = I_0 \, \mathrm{L}(\ddot{w}_0) \tag{22}$$

$$M_{xx,x} + M_{xy,y} - Q_x = I_2 \, \mathrm{L}(\ddot{\theta}_x) \tag{23}$$

$$M_{xy,x} + M_{yy,y} - Q_y = I_2 \, \mathrm{L}(\ddot{\theta}_y) \tag{24}$$

where

$$\mathrm{L}(N^{nl}) = (1 - \mu\nabla^2)\left\{ (N_{xx}^{nl}(w_0 + w^*)_{,x} + N_{xy}^{nl}(w_0 + w^*)_{,y})_{,x} + (N_{xy}^{nl}(w_0 + w^*)_{,x} + N_{yy}^{nl}(w_0 + w^*)_{,y})_{,y} \right\} \tag{25}$$

As it can be seen, the above equation will contain terms including nonlocal stress resultants and thus it cannot be readily reduced to a relation in terms of displacement field variables. To overcome this issue, Reddy [8] proposed to linearize the expression of N^{nl} as:

$$N^{nl} \approx N_{xx}^{nl}(w_0 + w^*)_{,xx} + 2N_{xy}^{nl}(w_0 + w^*)_{,xy} + N_{yy}^{nl}(w_0 + w^*)_{,yy} \tag{26}$$

Applying operator L to the above relation yields:

$$\mathrm{L}(N^{nl}) = N = N_{xx}(w_0 + w^*)_{,xx} + 2N_{xy}(w_0 + w^*)_{,xy} + N_{yy}(w_0 + w^*)_{,yy} \tag{27}$$

Now, by substitution from Eqs. (7-9) and Eq. (27) into Eqs. (20-24), the following governing equations in terms of displacement components will be obtained:

$$\begin{aligned} &A_{11}(u_{0,xx} + w_{0,x}w_{0,xx} + w_{0,xx}w^*_{,x} + w_{0,x}w^*_{,xx}) + A_{12}(v_{0,xy} + w_{0,y}w_{0,xy} + w_{0,xy}w^*_{,y} + w_{0,y}w^*_{,xy}) \\ &\quad + A_{66}(u_{0,yy} + v_{0,xy} + w_{0,xy}w_{0,y} + w_{0,x}w_{0,yy} + w_{0,xy}w^*_{,y} + w_{0,x}w^*_{,yy} + w_{0,yy}w^*_{,x} + w_{0,y}w^*_{,xy}) + \mathrm{L}(f_x) \\ &\qquad = I_0(\ddot{u}_0 - \mu\ddot{u}_{0,xx} - \mu\ddot{u}_{0,yy}) \end{aligned} \tag{28}$$

$$\begin{aligned} &A_{12}(u_{0,xy} + w_{0,x}w_{0,xy} + w_{0,xy}w^*_{,x} + w_{0,x}w^*_{,xy}) + A_{22}(v_{0,yy} + w_{0,y}w_{0,yy} + w_{0,yy}w^*_{,y} + w_{0,y}w^*_{,yy}) \\ &\quad + A_{66}(u_{0,xy} + v_{0,xx} + w_{0,xx}w_{0,y} + w_{0,x}w_{0,xy} + w_{0,xx}w^*_{,y} + w_{0,x}w^*_{,xy} + w_{0,xy}w^*_{,x} + w_{0,y}w^*_{,xx}) + \mathrm{L}(f_y) \\ &\qquad = I_0(\ddot{v}_0 - \mu\ddot{v}_{0,xx} - \mu\ddot{v}_{0,yy}) \end{aligned} \tag{29}$$

$$\begin{aligned} &K_s(A_{44}(w_{0,yy} + \theta_{y,y}) + A_{55}(w_{0,xx} + \theta_{x,x})) \\ &\quad + \left[A_{11}\left(u_{0,x} + \frac{1}{2}w_{0,x}^2 + w_{0,x}w^*_{,x}\right) + A_{12}\left(v_{0,y} + \frac{1}{2}w_{0,y}^2 + w_{0,y}w^*_{,y}\right)\right](w_{0,xx} + w^*_{,xx}) \\ &\quad + 2\left[A_{66}(u_{0,y} + v_{0,x} + w_{0,x}w_{0,y} + w_{0,x}w^*_{,y} + w_{0,y}w^*_{,x})\right](w_{0,xy} + w^*_{,xy}) \\ &\quad + \left[A_{12}\left(u_{0,x} + \frac{1}{2}w_{0,x}^2 + w_{0,x}w^*_{,x}\right) + A_{22}\left(v_{0,y} + \frac{1}{2}w_{0,y}^2 + w_{0,y}w^*_{,y}\right)\right](w_{0,yy} + w^*_{,yy}) + \mathrm{L}(q) \\ &\qquad = I_0\left(\ddot{w}_0 - \mu\ddot{w}_{0,xx} - \mu\ddot{w}_{0,yy}\right) \end{aligned} \tag{30}$$

$$D_{11}\theta_{x,xx} + D_{12}\theta_{y,xy} + D_{66}(\theta_{x,yy} + \theta_{y,xy}) - K_sA_{55}(w_{0,x} + \theta_x) = I_2(\ddot{\theta}_x - \mu\ddot{\theta}_{x,xx} - \mu\ddot{\theta}_{x,yy}) \tag{31}$$

$$D_{12}\theta_{x,xy} + D_{22}\theta_{y,yy} + D_{66}(\theta_{x,xy} + \theta_{y,xx}) - K_sA_{44}(w_{0,y} + \theta_y) = I_2(\ddot{\theta}_y - \mu\ddot{\theta}_{y,xx} - \mu\ddot{\theta}_{y,yy}) \tag{32}$$

These governing equations consist of five nonlinear coupled partial differential equations which takes into account the effects of initial geometric imperfection and nonlocal parameter, which should be solved simultaneously by considering appropriate boundary conditions. Classical governing equations could be obtained from the above relations by letting the nonlocal parameter μ be equal to zero. Also, the effects of geometric imperfection can be eliminated from the equations by taking $w^* = 0$.

4 SOLUTION PROCEDURE

4.1 Differential quadrature method

Differential quadrature method (DQM) is adopted here to discretize the governing equations and associated boundary conditions. Application of this method is suggested by many researchers for solving partial differential Eqs. (17) and (18). It is actually based on polynomial approximation of a function at any location in the domain. Then, partial derivatives of that function with respect to a domain coordinate are approximated by weighted summing of its values at all selected discrete points in the direction of that coordinate. Here, the method is briefly described for two-dimensional rectangular domain case [18]. Let $f_\alpha^{(n)}$ be the n th derivative of a function $f(x,y)$ with respect to α direction, where α may be x or y. According to DQ method:

$$f_x^{(1)}(x_i, y_j) = \sum_{k=1}^{N} a_{ik}^x \cdot f(x_k, y_j) \tag{33}$$

$$f_y^{(1)}(x_i, y_j) = \sum_{k=1}^{M} a_{jk}^y \cdot f(x_i, y_k) \qquad \text{for } \ i = 1,2,...,N; \qquad j = 1,2,...,M \tag{34}$$

where N and M are the number of sampling DQ grid points along x and y directions respectively. a_{ik}^x, a_{jk}^y are the corresponding weighting coefficients and are obtained from the following relations [18]:

$$a_{ij}^x = \frac{P^{(1)}(x_i)}{(x_i - x_j)P^{(1)}(x_j)} \qquad \text{for } \ j \neq i \tag{35}$$

$$a_{ii}^x = -\sum_{j=1, j\neq i}^{N} a_{ij}^x \qquad \text{for } \ i,j = 1,2,...,N \tag{36}$$

$$a_{ij}^y = \frac{Q^{(1)}(y_i)}{(y_i - y_j)Q^{(1)}(y_j)} \qquad \text{for } \ j \neq i \tag{37}$$

$$a_{ii}^y = -\sum_{j=1, j\neq i}^{M} a_{ij}^y \qquad \text{for } \ i,j = 1,2,...,M \tag{38}$$

where

$$P^{(1)}(x_i) = \prod_{j=1, j\neq i}^{N} (x_i - x_j), \qquad Q^{(1)}(y_i) = \prod_{j=1, j\neq i}^{M} (y_i - y_j) \tag{39}$$

Similarly, weighting coefficients for the second and higher derivatives are obtained as follows [18]:

$$c_{ij}^{(r)} = n\left[(a_{ij}^x \cdot c_{ii}^{(r-1)}) - \frac{c_{ij}^{(r-1)}}{x_i - x_j}\right] \qquad \text{for } \ j \neq i \tag{40}$$

$$c_{ii}^{(r)} = -\sum_{j=1, j\neq i}^{N} c_{ij}^{(r)} \qquad for\ \ r = 2,3,...,N-1;\ \ i,j = 1,2,...,N \tag{41}$$

$$\overline{c}_{ij}^{(s)} = n\left[(a_{ij}^{y}\cdot \overline{c}_{ii}^{(s-1)}) - \frac{\overline{c}_{ij}^{(s-1)}}{y_i - y_j}\right] \qquad \text{for}\ \ j\neq i \tag{42}$$

$$\overline{c}_{ii}^{(s)} = -\sum_{j=1, j\neq i}^{M} \overline{c}_{ij}^{(s)} \qquad \text{for}\ \ s = 2,3,...,M-1;\ \ i,j = 1,2,...,M \tag{43}$$

where $c_{ij}^{(r)}$ are the weighting coefficients of the *r*th order derivative of $f(x,y)$ with respect to *x* at (x_i, y_j), i.e. $f_x^{(r)}$, and $\overline{c}_{ij}^{(s)}$ are the weighting coefficients of the *s*th order derivative of $f(x,y)$ with respect to *y* at (x_i, y_j), i.e. $f_x^{(s)}$:

$$f_x^{(r)}(x_i, y_j) = \sum_{k=1}^{N} c_{ik}^{(r)} \cdot f(x_k, y_j) \tag{44}$$

$$f_y^{(s)}(x_i, y_j) = \sum_{k=1}^{M} \overline{c}_{jk}^{(s)} \cdot f(x_i, y_k) \quad \text{for}\ \ i = 1,2,...,N;\ \ j = 1,2,...,M;\ \ r = 1,2,...,N-1;\ \ m = 1,2,...,M-1 \tag{45}$$

At any location in the domain, $f(x,y)$ can be computed from

$$f(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{M} f(x_i, y_j)\cdot \eta_i(x)\cdot \gamma_j(y) \tag{46}$$

where $\eta_i(x)$ and $\gamma_j(y)$ in Eq. (46) are the Lagrange interpolation polynomials given by

$$\eta_i(x) = \prod_{k=1, k\neq i}^{N} \frac{x - x_k}{x_i - x_k} \tag{47}$$

$$\gamma_j(x) = \prod_{k=1, k\neq j}^{M} \frac{y - y_k}{y_j - y_k} \tag{48}$$

If the computational domain is normalized to $0 \le X \le 1,\ 0 \le Y \le 1$, then a Gauss-Lobatto-Chebyshev mesh generation might be taken for creating sampling grid points [18]:

$$X_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right], \qquad i = 1,2,\ldots,N \tag{49}$$

$$Y_j = \frac{1}{2}\left[1 - \cos\left(\frac{j-1}{M-1}\pi\right)\right], \qquad j = 1,2,\ldots,M \tag{50}$$

4.2 Derivation of Natural Frequencies

The following periodic solution is considered for field variables $(u, v, w, \theta_x, \theta_y)$:

$$\begin{Bmatrix} u_0(x,y,t) \\ v_0(x,y,t) \\ w_0(x,y,t) \\ \theta_x(x,y,t) \\ \theta_y(x,y,t) \end{Bmatrix} = \sum_{i=1}^{N}\sum_{j=1}^{M} \begin{Bmatrix} u_0(x_i,y_j) \\ v_0(x_i,y_j) \\ w_0(x_i,y_j) \\ \theta_x(x_i,y_j) \\ \theta_y(x_i,y_j) \end{Bmatrix} \eta_i(x)\gamma_j(y)\, \mathrm{e}^{i\omega t} \tag{51}$$

Applying DQM which was described in previous section, Eqs. (28-32) can be written as:

$$([K]_{5NM\times 5NM} - \omega^2 [M]_{5NM\times 5NM})\{X\}_{5NM\times 1} = \{0\} \tag{52}$$

where $[K]$ and $[M]$ are called stiffness and mass matrices respectively and ω is the natural angular frequency. The nodal displacement vector $\{X\}$ is defined as:

$$\{X\} = \left[\{X_1\} \quad \{X_2\} \quad \cdots \quad \{X_{N\times M}\}\right]^T \tag{53}$$

In which, $\{X_l\}$ is the displacement components vector of the lth node. Eq. (52) is a standard eigenvalue problem from which, after applying appropriate boundary conditions, the eigenvalues and eigenvectors of the mathematical problem, corresponding to natural frequencies and mode shapes of the physical structure, could be extracted.

5 NUMERICAL RESULTS AND DISCUSSION

In the following, free vibration results of clamped rectangular imperfect plates based on nonlocal elasticity theory are presented. Effects of non-local theory and imperfection parameter on the values of first natural frequency are indicated.

5.1 Verification of results

For the purpose of verification, free vibration analysis of a perfect plate without considering nonlocal effects is performed and the results are compared with those given by Leissa [19] in Table 1. A mesh size of 13×13 is used to derive the first five non-dimensional natural frequencies ($\lambda = \omega a^2 \sqrt{\rho h / D_{11}}$) of the plate for various side length ratios. Note that Leissa's results are given from the Rayleigh-Ritz. It can be observed from Table 1 that the results of current model agree very well with the data of Leissa.

Table 1
Comparison of non-dimensional natural frequencies of a perfect rectangular plate (local theory)

Mode Number	References	*a*/*b*				
		0.4	2/3	1.0	1.5	2.5
1	[19]	23.648	27.010	35.992	60.772	147.80
	Present	23.642	27.004	35.984	60.760	147.77
2	[19]	27.817	41.716	73.413	93.860	173.85
	Present	27.805	41.702	73.392	93.831	173.79
3	[19]	35.446	66.143	73.413	148.82	221.54
	Present	35.414	66.121	73.392	148.78	221.35
4	[19]	46.702	66.552	108.27	149.74	291.89
	Present	46.678	66.519	108.21	149.67	291.76
5	[19]	61.554	79.850	131.64	179.66	384.71
	Present	61.516	79.800	131.58	179.56	384.51

Table 2
Mechanical properties of graphene sheet [20]

Young's Modulus	1.02 *TPa*
Poisson Ratio	0.16
Density	2250 kg/m3

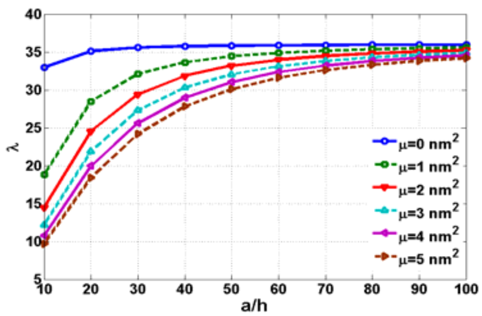

Fig. 2
Variation of non-dimensional frequency of a square grapheme sheet with *a/h* for different non-local parameter.

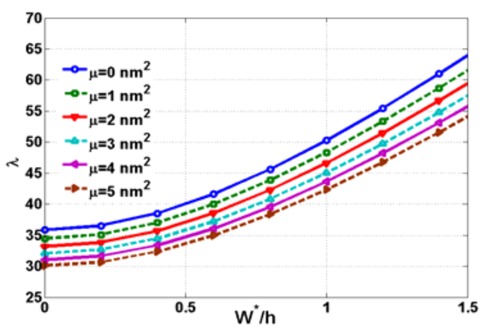

Fig. 3
Variation of non-dimensional frequency of a square grapheme sheet with W^* / h for different non-local parameter.

Discrepancies are observed because of differences of two theories adopted, i.e. first order shear deformation plate theory (FSDT) for current work and classical plate theory (CPT) for the work of Leissa. As it can be observed FSDT results in lower frequency parameters in comparison with CPT as expected.

5.2 Free Vibration of an imperfect graphene sheet

In this section, free vibration analysis of an imperfect square single layered graphene sheet is performed using the proposed DQ method. Geometric imperfection is taken as a sinusoidal wave of the following form [14]:

$$w^* = W^* \sin(\pi x / a)\sin(\pi y / b) \tag{54}$$

Table 2 presents mechanical properties of graphene material. Thickness of the sheet is taken as $h = 0.34$ nm [20]. Variation of non-dimensional first natural frequency, λ, versus length to height ratio, *a/h,* for various values of non-local parameter is shown in Fig. 2. It may be seen that for a specified value of a/h, frequency parameter, λ, decreases with increasing non-local parameter, μ, although this reduction in λ value becomes negligible for larger length to height ratios and the frequency parameter converges to the value for $\mu = 0$. This trend shows that non-local effects attenuate by increasing the characteristic geometric length of the structure. Furthermore, increasing length to thickness ratio results in increasing frequency parameter for all values of μ. This is mainly due to decrease in shear effects for greater values of *a/h*. Fig. 3 shows the effect of imperfection on frequency parameter of a grapheme sheet for various values of non-local parameter. It can be seen that as the amplitude of imperfection is increased, the frequency parameter, λ increases. The reason is that imperfection increases resistance of the plate against vibration. Besides, for a specified value of W^* / h, with increasing non-local parameter, μ, frequency parameter decreases.

6 CONCLUSION

Equations of motion based on nonlocal differential constitutive equations of Eringen and the von Kármán nonlinear strain-displacement relationship were obtained for the shear deformable nano-plates with initial geometric imperfections. Difficulties arising from nonlinear terms due to employment of nonlocal theory were removed by an approximation made to the nonlinear terms. With this simplification five coupled nonlinear partial differential equations in terms of displacement components were obtained. The governing equations are solved using differential quadrature method (DQM) and numerical results for free vibration of an imperfect single layered grapheme sheet are presented which show a significant effect of nonlocal and imperfection parameters on the natural frequency of the grapheme sheets.

REFERENCES

[1] Terrones M., 2010, Graphene and graphite nano ribbons: Morphology, properties, synthesis, defects and applications, *Nano Today* **5**: 351-372.

[2] Ke C.H., Pugnol N., Peng B., Espinosa H.D., 2005, Experiments and modeling of carbon nanotube-based NEMS devices, *Journal of the Mechanics and Physics of Solids* **53**: 1314-1333.

[3] Hierold C.h., Jungen A., Stampfer C.h., Helbling T.H., 2007, Nano electromechanical sensors based on carbon nanotubes, *Sensors and Actuators A* **136**: 51-61.

[4] Wang C.M., Tan V.B.C., Zhang Y.Y., 2006, Timoshenko beam model for vibration analysis of multi-walled carbon nanotubes, *Journal of Sound and Vibration* **294**: 1060-1072.

[5] Wang Q., Varadan V.K., 2006, Vibrationofcarbonnanotubesstudiedusingnonlocalcontinuummechanics, *Smart Materials and Structures* **15**: 659-666.

[6] He X.Q., Kitipornchai S., Liew K.M., 2005, Resonance analysis of multi-layered grapheme sheets used as nano scale resonators, *Nanotechnology* **16**: 2086-2091.

[7] Gibson R.F., Ayorinde O.E., Wen Y.F., 2007, Vibration of carbon nanotubes and there composites: a review, *Composites Science and Technology* **67**: 1-28.

[8] Reddy J.N., 2010, Nonlocal nonlinear formulations for bending of classical and shear deformation theories of beams and plates, *International Journal of Engineering Science*, in press.

[9] Reddy J.N., 2007, Nonlocal theories for bending, buckling, and vibration of beams, *International Journal of Engineering Science* **45**: 288-307.

[10] Pradhan S.C., Phadikar J.K., 2009, Nonlocal elasticity theory for vibration of nano plates, *Journal of Sound and Vibration* **325**: 206-223.

[11] Murmu T., Pradhan S.C., 2009, Small-scale effect on the free in-plane vibration of nano plates by nonlocal continuum model, *Physica E: Low-dimensional Systems and Nanostructures* **41:**1628-1633.

[12] Behfar K., Naghdabadi R., 2005, Nanoscale vibrational analysis of a multi-layered grapheme sheet embedded in an elastic medium, *Composites Science and Technology* **65**: 1159-1164.

[13] Liew K.M., He X.Q., Kitipornchai S., 2006, Predicting nano vibration of multi-layered grapheme sheets embedded in an elastic matrix, *Acta Materialia* **54**: 4229-4236.

[14] Kitipornchai S., Yang J., Liew K.M., 2004, Semi-analytical solution for nonlinear vibration of laminated FGM plates with geometric imperfections, *International Journal of Solids and Structures* **41**: 2235-2257.

[15] Yang J., Shen H.S., 2002, Vibration characteristics and transient response of shear deformable functionally graded plates in thermal environment, *Journal of Sound and Vibration* **255**: 579-602.

[16] Eringen A.C., 2002, *Nonlocal Continuum Field Theories*, Springer, New York.

[17] Pradhan, S.C., Murmu T.,2009, Small scale effect on the buckling of single-layered grapheme sheets under bi-axial compression via nonlocal continuum mechanics, *Computational Materials Science* **47**(1): 268-274.

[18] Shu C., 2000, *Differential Quadrature and Its Application in Engineering*, Springer, London.

[19] Leissa, A.W., 1973, The free vibration of rectangular plates, *Journal of Sound and Vibration* **31**(3): 257-293.

[20] Kitipornchai S., He X.Q, Liew K.M., 2005, Continuum model for the vibration of multilayered grapheme sheets. *Physical Review B* **72**: 6, (075443).