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Study Of Thermoelastic Damping in an Electrostatically Deflected Circular Micro-Plate Using Hyperbolic Heat Conduction Model

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ABSTRACT

Thermoelastic damping (TED) in a circular micro-plate resonator subjected to an electrostatic pressure is studied. The coupled thermo-elastic equations of a capacitive circular micro plate are derived considering hyperbolic heat conduction model and solved by applying Galerkin discretization method. Applying complex-frequency approach to the coupled thermo-elastic equations, TED is obtained for different ambient temperatures. Effects of the geometrical parameters on TED and the critical thickness are investigated. Furthermore, the effect of applied bias DC voltage on TED for an electrostatically deflected micro-plate is also investigated.

Keywords: MEMS; Internal damping; Quality factor; Circular Micro-Plate; Electrostatic actuation

1 INTRODUCTION

MICRO-ELECTROMECHANICAL systems (MEMS) technology has been rapidly growing since its beginnings in the early 1980's. As MEMS are light, small and consume few energy, they are used in a wide spectrum of areas of engineering such as aerospace, medical, automotive or information technology, ink jet printer heads, micropumps, projection display arrays and airbag accelerometers [1]. In the development of MEMS systems, it is necessary to know how the parameters affect their physical and mechanical behaviors [2], [3]. One of these parameters is damping. For instance, in micro-accelerometers, high damping (near critical) is desirable to prevent large resonance responses due to external disturbances, which might result in a mechanical or an electrical failure. On the other hand, in resonant sensors, a low damping is required for achieving high sensitivity and resolution and low power consumption. Hence, it is essential to understand the damping mechanisms in MEMS devices to optimize their designs [4].

Traditionally, the dissipation level in MEMS is measured by means of the so-called quality factor (Q), which is proportional to the ratio between the resonant frequency for the undamped system and the damping factor. If the damping is not excessive (i.e. is larger than a certain threshold), the inverse of the quality factor can be physically interpreted as the fraction of energy lost per radian [5]. Many researchers have discussed different damping mechanisms in MEMS, such as doping-impurities losses, support-related losses, thermoelastic damping and the Akhiezer effect, as well as the radiation of energy away from the resonator into its surroundings [2]. Many of the

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damping mechanism can be minimized by optimizing the design. After optimizing the design the TED may be the main effect, limiting the quality factor (Q) in mechanical resonators vibrating under vacuum [6]. Losses due to bulk defects and impurities, and losses due to the thermoelastic effect and other phonon scattering phenomena are some of the internal Damping [7]. Over the past decade, both experimental and theoretical studies [1]-[19], have highlighted the important role of TED in micromechanical resonators. Zener [18], [19] firstly identified the existence of TED as a significant dissipation mechanism in flexural resonators. Roszhart observed TED in single-crystal silicon micro-resonators at room temperature. Yasumura et al. also reported thermoelastic damping in silicon nitride micro-resonators at room temperature, and their measured results are an order of magnitude smaller than Roszhart's. Houston et al. studied the importance of thermoelastic damping for silicon-based MEMS. Their results indicate that the TED arising from this mechanism is strong and persists down to 50nm scale structures [3]. Lifshitz studied thermoelastic damping of a beam with rectangular cross-sections [12] and Phonon-mediated dissipation in micro- and Nano-mechanical systems [21]. Nayfeh and Younis present a model and analytical expressions for the quality factors of micro-plates of general shapes and boundary conditions due to TED [4]. Thermoelastic Damping can be caused by the transversal heat flow in bending beams\plate from compressed to extended regions or due to heat exchanges with the environment. The transversal heat flow due to the thermoelastic effect is more dependent on intrinsic physical properties than on the structure and is probably the only effect which exactly produces a Debye peak [22]. Rezazadeh et al have calculated the Quality factor of micro-beam resonators with bias DC voltage [1] and Armin S. Vahdat et al has been evaluated the TED in micro-beam resonators with axial and residual stresses [23] and with Piezoelectric Layers [24]. Modeling and simulation of TED mechanisms is a recurrent interest in the community of micromechanics and micro-engineering, mainly motivated by the recent advancement of MEMS and NEMS technologies and the associated energy dissipation problems in high performance actuators, resonators and filters [25].

Today because of advantages of electrostatic MEM actuators, such as, favorable scaling property, low energy consumption, low cost, low driving power, large deflection capacity, relative ease of fabrication and others, electrostatic MEM actuators are widely used in micro-structure. Therefore, electrostatic MEM actuators are one of the most common and important actuators in MEMS [26]. Micro-plates (esp. circular micro plates) have been increasingly tested and used in various MEMS devices [26-30], such as micropumps and pressure sensors. Nayfeh and Younis [4] investigated the TED in an electrostatically actuated micro-beam and rectangular micro- plate considering classical Fourier or parabolic heat conduction model. But the influence of using hyperbolic heat conduction model instead of parabolic model in TED is still unknown in their investigation. Their results were upon frequency shift ratio and they did not discuss about TED sufficiently and they did not consider electrostatic actuation in their studies.

According to the previous works as discussed, TED for micro-structures under electrostatic loading considering hyperbolic heat conduction model when they are subjected to a bias DC voltage, which decreases the equivalent stiffness of the structure especially near the pull-in voltage, is not studied sufficiently. Hence, in this paper an electrostatically deflected tunable circular micro-plate resonator is considered, and TED in small oscillations about an electrostatically deflected position is studied using hyperbolic heat conduction model. Effects of the plate thickness and radius, ambient temperature and applied bias DC voltages on TED ratio are studied using Galerkin discretization method and complex-frequency approach. In addition, the value of the critical thickness is determined and effects of the ambient temperature and radius on it are studied.

2 MODEL DESCRIPTION

We consider a micro-plate Fig. 1 actuated by an electrostatic load with radius of R and the thickness of h , with Young's modulus E and Poisson's ratio v . Since the micro plate resonator is assumed to be subjected to a bias DC voltage, an electrostatic force can be represented as follows [1, 31]:

$$F(w,V)=\frac{\varepsilon_0 V^2}{2(g_0-w(r,t))^2} \tag{1}$$

where ε_0 is the dielectric (permittivity) of the air, V is the applied voltage and g_0 is the initial gap between the movable micro-beam and the ground electrode, and $w(r,t)$ is the deflection of the micro-beam, defined to be positive downward. In order to study TED in an electrostatically deflected micro plate, a bias DC voltage is applied to the micro-plate and small free vibrations of the micro-plate about the static equilibrium position are studied.

3 GOVERNING EQUATIONS

Since the proposed micro-plate thickness is very lesser than its diameter, transverse shear deformation is negligible and micro-plate motion can be modeled based on Kirchhoff classic thin plate Theory [32-33]. It is considered that the micro-plate is actuated by an electrostatic load and assumed that all forms of extrinsic damping, such as viscous damping, can be neglected. By introducing $u(r,\theta,z,t)$, $v(r,\theta,z,t)$ and $w(r,\theta,z,t)$ as displacement components along the radial, circumferential and axial directions, respectively, and $T(r,\theta,z,t)$ the temperature field the strain and stress components can be written as below:

3.1 Strain components

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\theta = \frac{u}{r} + \frac{\partial v}{r\partial\theta}, \qquad \gamma_{r\theta} = \frac{\partial u}{r\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \tag{2}$$

$$u = -z\frac{\partial w}{\partial r}, \qquad v = -z\frac{1}{r}\frac{\partial w}{\partial\theta} \tag{3}$$

3.2 Stress components

$$\sigma_r = \frac{E}{1-\upsilon^2}\left[(\varepsilon_r + \upsilon\varepsilon_\theta) - (1+\upsilon)\alpha_T T\right] \tag{4}$$

$$\sigma_\theta = \frac{E}{1-\upsilon^2}\left[(\varepsilon_\theta + \upsilon\varepsilon_r) - (1+\upsilon)\alpha_T T\right] \tag{5}$$

$$\tau_{r\theta} = G\gamma_{r\theta} = 2G\frac{\partial v}{\partial r} \tag{6}$$

By replacing Eq. (9) in Eq. (8) we can write the trace of strain tensor (ε_{ii}) as below

$$\varepsilon_{ii} = \varepsilon_r + \varepsilon_\theta = -z\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{r\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) \tag{7}$$

where G is the shear modulus and T is the temperature changes, measured with respect to reference temperature T_0 and v is the position ratio and the Laplace operator is.

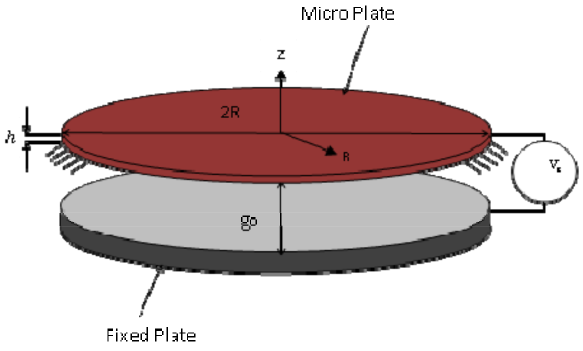

Fig. 1
Scheme of the micro-plate.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{8}$$

Circular plates are capable of both in-plane and out-of-plane vibrations, but in this paper only out-of-plane vibrations are considered. By assuming small strains and displacements, we obtain the following thermoelastic linear equation of motion governing the transverse deflection of the circular micro-plate with electrostatic load [3, 33]:

$$D\nabla^2\nabla^2 w + D(1+\upsilon)\alpha_T \nabla^2 M_T + \rho h \frac{\partial^2 w}{\partial t^2} = \frac{\varepsilon_0 V^2}{2(g_0 - w(r))^2} \tag{9}$$

where ρ is the density, α_T is the coefficient of thermal expansion. Note that in axisymmetric circular plate, the displacement and thermal moment are independent of θ so the Laplace operator is:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{10}$$

By introducing the thermal moment:

$$M_T = \frac{12}{h^3}\int_{-h/2}^{h/2} T(r,z,t) z \, dz \tag{11}$$

The plate flexural rigidity is:

$$D = \frac{Eh^3}{12(1-\upsilon^2)} \tag{12}$$

The thermal conduction equation containing the thermoelastic coupling term has the following form [2]:

$$k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) = \rho c_v \frac{\partial T}{\partial t} + \beta T_0 \frac{\partial \varepsilon_{ii}}{\partial t} \tag{13}$$

where k is the thermal conductivity, c_v is the specific heat at constant volume and the thermal modulus is $\beta = E\alpha_T / (1-2\upsilon)$.

Substituting Eq. (7) into Eq. (13) gives the thermal conduction equation for the plate

$$k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) = \rho c_v \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2}\right) + \beta T_0 \left(\dot{\varepsilon}_{ii} + \tau_0 \ddot{\varepsilon}_{ii}\right) \tag{14}$$

As discussed, before this paper only considers the axisymmetric problem for a circular plate. Noting that thermal gradients along the plate thickness direction are much larger than gradients along the radial direction, so Eq. (14) can be rewritten as below:

$$k\frac{\partial^2 T}{\partial z^2} - \rho c_v \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2}\right) - \beta T_0 \left(\dot{\varepsilon}_{ii} + \tau_0 \ddot{\varepsilon}_{ii}\right) = 0 \tag{15}$$

Often in MEMS a tuning DC voltage is used to decrease resonant frequency of micro-plate resonators and shift their band width. Applying a bias DC voltage leads to an initial static deflection, which can be expressed as $w_s(r)$.

Free vibration about the static equilibrium position can be studied by introducing the dynamic deflection of the micro-plate about the static equilibrium position as $w_d(r,t)$; total deflection can be expressed as:

$$w(r,t) = w_d(r,t) + w_s(r) \tag{16}$$

Substituting Eq. (14) in Eq. (1)

$$D\nabla^2\nabla^2(w_d(r,t)+w_s(r)) + D(1+\upsilon)\alpha_T\nabla^2 M_T + \rho h\frac{\partial^2(w_d(r,t)+w_s(r))}{\partial t^2} = \frac{\varepsilon_0 V^2}{2(g_0 - w_d(r,t) - w_s(r))^2} \tag{17}$$

Considering $w_d \ll (g_0 - w_s(r))$, the nonlinear electrostatic force $F(w,v)$ using Calculus of Variation Theory and Taylor series expansions can be written as [1,31]:

$$F(w_d + w_s) = \frac{\varepsilon_0 V^2}{2(g_0 - w_s)^2} + \frac{\varepsilon_0 V^2}{(g_0 - w_s)^3} w_d \tag{18}$$

So, the nonlinear equation of static deflection and equation of small oscillations of the electro statically deflected micro-plate can be written as below respectively:

$$D\nabla^2\nabla^2 w_s(r) = \frac{\varepsilon_0 V^2}{(g_0 - w_s(r))^{`2}} \tag{19}$$

$$D\nabla^2\nabla^2 w_d(r,t) + D(1+\upsilon)\alpha_T\nabla^2 M_T + \rho h\frac{\partial^2 w_d(r,t)}{\partial t^2} = \frac{\varepsilon_0 V^2}{(g_0 - w_s(r))^3} w_d(r,t) \tag{20}$$

For convenience, following non dimensional parameters are introduced to transform Eqs. (11, 15, 19, 20) into non dimensional forms:

$$\tilde{w}_s = \frac{w_s}{g_0},\ \tilde{w}_d = \frac{w_d}{h},\ \tilde{r} = \frac{r}{R},\ \tilde{z} = \frac{z}{h},\ \tilde{T} = \frac{T}{T_o},\ \tilde{t} = \frac{t}{t^*},\ t^* = R^2\sqrt{\frac{\rho h}{D}},\ \tilde{M}_T = M_T\frac{R^2}{h}$$

By substituting introduced non dimensional parameters into Eqs. (11,15,19,20) we have:

$$\tilde{M}_T = \frac{12R^2}{h^2}\int_{-1/2}^{1/2} \tilde{T}(\tilde{r},\tilde{z},\tilde{t})\tilde{z}\ \mathrm{d}\tilde{z} \tag{21}$$

The nonlinear equation of static deflection

$$\tilde{r}^3\frac{\partial^4\tilde{w}_s}{\partial\tilde{r}^4} + \tilde{r}^2\frac{\partial^3\tilde{w}_s}{\partial\tilde{r}^3} - \tilde{r}\frac{\partial^2\tilde{w}_s}{\partial\tilde{r}^2} + \frac{\partial\tilde{w}_s}{\partial\tilde{r}} = S_6\frac{V_i^2\tilde{r}^3}{(1-\tilde{w}_s)^2} \tag{22}$$

The nonlinear equation of small oscillations of the electro statically deflected micro-plate

$$\tilde{r}^3\frac{\partial^4\tilde{w}_d}{\partial\tilde{r}^4} + \tilde{r}^2\frac{\partial^3\tilde{w}_d}{\partial\tilde{r}^3} - \tilde{r}\frac{\partial^2\tilde{w}_d}{\partial\tilde{r}^2} + \frac{\partial\tilde{w}_d}{\partial\tilde{r}} + \tilde{r}^3 S_1\nabla^2\tilde{M}_T + \tilde{r}^3\frac{\partial^2\tilde{w}_d}{\partial\tilde{t}^2} = S_2\tilde{r}^3\frac{\tilde{w}_d(r,t)}{(1-\tilde{w}_s(r))^3} \tag{23}$$

and thermal conduction equation containing the thermo elastic coupling term

$$\tilde{r}\frac{kT_0}{h^2}\frac{\partial^2\tilde{T}}{\partial\tilde{z}^2}-\rho c_v\left(\tilde{r}\frac{T_0}{t^*}\dot{\tilde{T}}+\tilde{r}\tau_0\frac{T_0}{t^{*2}}\ddot{\tilde{T}}\right)+\frac{\beta T_0 h}{R^2}\tilde{z}\left(\tilde{r}\frac{g_0}{t^*}\nabla^2\dot{\tilde{w}}+\tilde{r}\tau_0\frac{g_0}{t^{*2}}\nabla^2\ddot{\tilde{w}}\right)=0 \tag{24}$$

and

$$\tilde{r}\frac{\partial^2\tilde{T}}{\partial\tilde{z}^2}-D_1\tilde{r}\dot{\tilde{T}}+D_2\tilde{r}\ddot{\tilde{T}}+D_3\tilde{z}\left(\tilde{r}\nabla^2\dot{\tilde{w}}\right)+D_4\tilde{z}\left(\tilde{r}\nabla^2\ddot{\tilde{w}}\right)=0 \tag{25}$$

where

$$S_1=\frac{12(1+\upsilon)\alpha_T R^2}{h^2},\quad S_2=\frac{hR^4\varepsilon_0 V_{dc}^2}{D{g_0}^3},\quad D_1=\frac{\rho c_v h^2}{t^* k},\quad D_2=\frac{\tau_0}{t^*}D_1,\ D_3=\frac{\beta h^3}{kR^2}\frac{g_0}{t^*},\ D_4=\frac{\tau_0}{t^*}D_3 \tag{26}$$

4 NUMERICAL SOLUTION

4.1 Nonlinear equation of static deflection

Assuming that $\tilde{w}_s^i$ is the static deflection of the micro-plate due to the applied DC voltages V_i. By increasing the applied voltages to

$$V_{i+1}\to V_i+\Delta V,\qquad \tilde{w}_s^{i+1}\to\tilde{w}_s^i+\varpi_s(\tilde{r}) \tag{27}$$

By substituting Eq. (27) in Eq. (22) we have:

$$\tilde{r}^3\frac{\partial^4(\tilde{w}_s^i+\varpi_s)}{\partial\tilde{r}^4}+\tilde{r}^2\frac{\partial^3(\tilde{w}_s^i+\varpi_s)}{\partial\tilde{r}^3}-\tilde{r}\frac{\partial^2(\tilde{w}_s^i+\varpi_s)}{\partial\tilde{r}^2}+\frac{\partial(\tilde{w}_s^i+\varpi_s)}{\partial\tilde{r}}=S_6\frac{V^2\tilde{r}^3}{(1-\tilde{w}_s^i-\varpi_s)^2} \tag{28}$$

Using Calculus of Variation Theory and Taylor series expansion about $\tilde{w}_s^i$, and truncating its higher orders Eq. (28) can be written as follows:

$$\tilde{r}^3\frac{\partial^4(\tilde{w}_s^i+\varpi_s)}{\partial\tilde{r}^4}+\tilde{r}^2\frac{\partial^3(\tilde{w}_s^i+\varpi_s)}{\partial\tilde{r}^3}-\tilde{r}\frac{\partial^2(\tilde{w}_s^i+\varpi_s)}{\partial\tilde{r}^2}+\frac{\partial(\tilde{w}_s^i+\varpi_s)}{\partial\tilde{r}}=S_6(\frac{\tilde{r}^3V_{i+1}^2}{\left(1-\tilde{w}_s^i\right)^2}+2\frac{\tilde{r}^3V_{i+1}^2}{\left(1-\tilde{w}_s^i\right)^3}\varpi_s \tag{29}$$

Deducting Eq. (22) from Eq. (29) leads to the following equation:

$$\tilde{r}^3\frac{\partial^4\varpi_s}{\partial\tilde{r}^4}+\tilde{r}^2\frac{\partial^3\varpi_s}{\partial\tilde{r}^3}-\tilde{r}\frac{\partial^2\varpi_s}{\partial\tilde{r}^2}+\frac{\partial\varpi_s}{\partial\tilde{r}}-2S_6\frac{\tilde{r}^3V_{i+1}^2}{(1-\tilde{w}_s^i)^3}\varpi_s=S_6\frac{\tilde{r}^3(V_{i+1}^2-V_i^2)}{(1-\tilde{w}_s^i)^2} \tag{30}$$

Galerkin method is applied in each step to calculate ϖ_s. Choosing suitable shape function $\Theta_j(\tilde{r})$ satisfying geometrical boundary conditions of the micro-plate ϖ_s can be approximated by following series:

$$\varpi_s(\tilde{r})=\sum_{j=1}^{q}\Theta_j(\tilde{r}) \tag{31}$$

Increasing the applied DC voltage and calculating the deflection in each step, the deflection can be calculated at any given bias DC voltage.

4.2 Equation of small oscillations coupled with thermal conduction equation

To calculate the effects of thermoelastic coupling on the vibrations of a circular plate, we solve the coupled thermoelastic Eq. (23) and Eq. (24) by applying a Galerkin based reduced order model and determine TED using complex-frequency approach. These equations have an evident dependency on the static deflection of the micro-plate. Thus, it is necessary to solve the nonlinear static Eq. (23) at the first [31]. Using a Galerkin based reduced order model approximate solutions of dynamic motion and heat distribution of the micro-plate can be given as following respectively:

$$\tilde{w}_d(\tilde{r},\tilde{t}) = \sum_{k=1}^{p} \varphi_k(\tilde{r})\psi_k(\tilde{t}) \tag{32}$$

$$\tilde{T}(\tilde{r},\tilde{z},\tilde{t}) = \sum_{i=1}^{m}\sum_{j=1}^{n} \xi_i(\tilde{r})\zeta_j(\tilde{z})\chi_{ij}(\tilde{t}) \tag{33}$$

By substituting Eq. (33) into Eq. (21) thermal moment equation can be written:

$$\tilde{M}_T = \frac{12R^2}{h^2}\int_{1/2}^{1/2} \tilde{T}\tilde{z}\ \mathrm{d}\tilde{z} = \frac{12R^2}{h^2}\sum_{i=1}^{m}\sum_{j=1}^{n} \xi_i(\tilde{r})\chi_{ij}(\tilde{t}) = \int_{-1/2}^{1/2} \zeta_j\left(\tilde{z}\right)\tilde{z}\ \mathrm{d}\tilde{z} \tag{34}$$

and also substituting Eqs. (32-34) into Eq. (28) and Eq. (29) leads to the following equations:

$$\tilde{M}_T = \frac{12R^2}{h^2}\int_{-1/2}^{1/2} \tilde{T}\tilde{z}\ \mathrm{d}\tilde{z} = \frac{12R^2}{h^2}\sum_{i=1}^{m}\sum_{j=1}^{n} \xi_i(\tilde{r})\chi_{ij}(\tilde{t}) = \int_{-1/2}^{1/2} \zeta_j(\tilde{z})\tilde{z}\ \mathrm{d}\tilde{z} \tag{35}$$

and

$$\begin{aligned} &\tilde{r}\sum_{i=1}^{m}\sum_{j=1}^{n} \xi_i(\tilde{r})\zeta_j^{(II)}(\tilde{z})\chi_{ij}(\tilde{t}) - D_1\tilde{r}\sum_{i=1}^{m}\sum_{j=1}^{n} \xi_i(\tilde{r})\zeta_j(\tilde{z})\dot{\chi}_{ij}(\tilde{t}) - D_2\tilde{r}\sum_{i=1}^{m}\sum_{j=1}^{n} \xi_i(\tilde{r})\zeta_j(\tilde{z})\ddot{\chi}_{ij}(\tilde{t}) \\ &\quad + D_3\tilde{z}\sum_{k=1}^{p}(\tilde{r}\varphi_k^{(II)}(\tilde{r}) + \varphi_k^{(I)}(\tilde{r}))\dot{\psi}_k(\tilde{r}) + D_4\tilde{z}\sum_{k=1}^{p}(\tilde{r}\varphi_k^{(II)}(\tilde{r}) + \varphi_k^{(I)}(\tilde{r}))\ddot{\psi}_k(\tilde{t}) = \epsilon_2 \end{aligned} \tag{36}$$

According to Galerkin method, following conditions should be satisfied:

$$\int_0^1 \varphi_f(\tilde{r})\epsilon_1\ \mathrm{d}\tilde{r} = 0, \qquad f = 1,\ldots,p \tag{37}$$

$$\int_0^1\int_{-1/2}^{1/2} \zeta_h(\tilde{z})\xi_g(\tilde{r})\epsilon_2\ \mathrm{d}\tilde{z}\ \mathrm{d}\tilde{r} = 0, \qquad g = 1,\ldots,m, \qquad h = 1,\ldots,n \tag{38}$$

and by applying Eqs. (34), (35) to Eqs. (32), (33) we have the following equations:

$$\begin{aligned} &\sum_{k=1}^{p} K_{fk}^{(1)}\psi_k + \sum_{k=1}^{p} K_{fk}^{(2)}\psi_k - \sum_{k=1}^{p} K_{fk}^{(3)}\psi_k + \sum_{k=1}^{p} K_{fk}^{(4)}\psi_k + S_1\sum_{i=1}^{m}\sum_{j=1}^{n}(K_{fi}^{(5)} + K_{fi}^{(6)})K_j^{(7)}\chi_{ij} \\ &\quad + \sum_{k=1}^{p} K_{fk}^{(8)}\ddot{\psi}_k - S_2\sum_{k=1}^{p} K_{fk}^{(9)}\psi_k = 0 \end{aligned} \tag{39}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n}G_{gi}^{(1)}G_{hj}^{(2)}\chi_{ij}(\tilde{t})-D_1\tilde{r}\sum_{i=1}^{m}\sum_{j=1}^{n}G_{gi}^{(3)}G_{hj}^{(4)}\dot{\chi}_{ij}(\tilde{t})-D_2\tilde{r}\sum_{i=1}^{m}\sum_{j=1}^{n}G_{gi}^{(5)}G_{hj}^{(6)}\ddot{\chi}_{ij}(\tilde{t})+D_3\sum_{k=1}^{p}(G_{gk}^{(7)} \\ +G_{gk}^{(8)})G_{h}^{(9)}\dot{\psi}_k(\tilde{t})+D_4\sum_{k=1}^{p}(G_{gk}^{(10)}+G_{gk}^{(11)})G_{h}^{(12)}\ddot{\psi}_k(\tilde{t})=0 \tag{40}$$

in which:

$$\begin{aligned}
&K_{fk}^{(1)}=\int_0^1\varphi_f\varphi_k{}^{(IV)}\tilde{r}^3\,\mathrm{d}\tilde{r},\qquad K_{fk}^{(2)}=\int_0^1\varphi_f\varphi_k{}^{(III)}\tilde{r}^2\,\mathrm{d}\tilde{r},\qquad K_{fk}^{(3)}=\int_0^1\varphi_f\varphi_k{}^{(II)}\tilde{r}\,\mathrm{d}\tilde{r}\\
&\mathrm{K}_{\mathrm{fk}}^{(4)}=\int_0^1\varphi_f\varphi_k{}^{(\mathrm{I})}\mathrm{d}\tilde{r},\qquad \mathrm{K}_{\mathrm{fi}}^{(5)}=\int_0^1\varphi_f\xi_i{}^{(\mathrm{II})}\tilde{r}^3\mathrm{d}\tilde{r},\qquad \mathrm{K}_{\mathrm{fi}}^{(6)}=\int_0^1\varphi_f\xi_i{}^{(\mathrm{I})}\tilde{r}^2\mathrm{d}\tilde{r}\\
&\mathrm{K}_{\mathrm{j}}^{(7)}=\int_{-1/2}^{1/2}\tilde{z}\zeta_j\mathrm{d}\tilde{z},\qquad \mathrm{K}_{\mathrm{fk}}^{(8)}=\int_0^1\tilde{r}^3\varphi_f\varphi_k\mathrm{d}\tilde{r},\qquad \mathrm{K}_{\mathrm{fk}}^{(9)}=\int_0^1\tilde{r}^3\frac{\varphi_f\varphi_k}{(1-\tilde{w}_s)^3}\mathrm{d}\tilde{r}\\
&\mathrm{G}_{\mathrm{gi}}^{(1)}=\mathrm{G}_{\mathrm{gi}}^{(3)}=\mathrm{G}_{\mathrm{gi}}^{(5)}=\int_0^1\tilde{r}\xi_g\xi_i\mathrm{d}\tilde{r},\qquad \mathrm{G}_{\mathrm{hj}}^{(2)}=\int_0^1\zeta_h\zeta_j{}^{(\mathrm{II})}\mathrm{d}\tilde{z},\qquad \mathrm{G}_{\mathrm{hj}}^{(4)}=\mathrm{G}_{\mathrm{hj}}^{(6)}=\int_0^1\zeta_h\zeta_j\mathrm{d}\tilde{z}\\
&\mathrm{G}_{\mathrm{gk}}^{(7)}=\mathrm{G}_{\mathrm{gk}}^{(10)}=\int_0^1\xi_g\varphi_k{}^{(\mathrm{II})}\tilde{r}\mathrm{d}\tilde{r},\qquad \mathrm{G}_{\mathrm{gk}}^{(8)}=\mathrm{G}_{\mathrm{gk}}^{(11)}=\int_0^1\xi_g\varphi_k{}^{(\mathrm{I})}\mathrm{d}\tilde{r},\qquad G_h^{(9)}=G_h^{(12)}=\int_0^1\zeta_h\tilde{z}\,\mathrm{d}\tilde{z}
\end{aligned} \tag{41}$$

Choosing suitable shape functions in Eq. (32) and Eq. (33) which satisfy the boundary conditions, and solving Eq. (40) and Eq. (41) simultaneously in which $\psi_k(\tilde{t})=\mathrm{e}^{i\omega_k\tilde{t}}$ and $\chi_{ij}(\tilde{t})=\mathrm{e}^{i\omega_{ij}\tilde{t}}$, complex frequencies are achieved. Note that $w(\tilde{r},\tilde{t})$ and $\tilde{T}(\tilde{r},\tilde{z},\tilde{t})$ vibrate at the same frequency, therefore for the first mode analysis it is considered $\omega_k=\omega_{ij}=\omega$ According to complex frequency approach, the thermoelastic damping expressed in terms of the inverse of the quality factor is then given by [20]:

$$\int_0^1\varphi_f(\tilde{r})\epsilon_1\,\mathrm{d}\tilde{r}=0,\qquad f=1,\dots,p \tag{42}$$

where Re (ω) is the real part and Im (ω) is the imaginary part of the complex frequency.

5 NUMERICAL RESULTS

The considered micro-plate in this study has the material and geometrical properties as given in Table 1. The coefficient of linear thermal expansion has a temperature and size dependent nature but in the present study it is assumed to be constant for all temperatures.

Table 1
Mechanical and thermal properties of silicon under different temperatures [34-35].

$T_\circ$	E (GPa)	ρ (kgm^{-3})	υ	k $(Wm^{-1}K^{-1})$	c_v $(Jkg^{-1}K^{-1})$	α_T $(10^{-6}K^{-1})$
200	166.0	2330	0.22	266	557	1.406
240	165.9	2330	0.22	222	624.1	1.986
300	165.9	2330	0.22	156	718	2.616

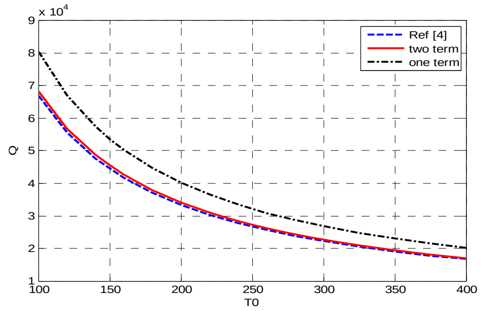

Fig. 2
Comparison of Q for different temperatures, $h = 10\ \mu m$ and *R*/*h*=50.

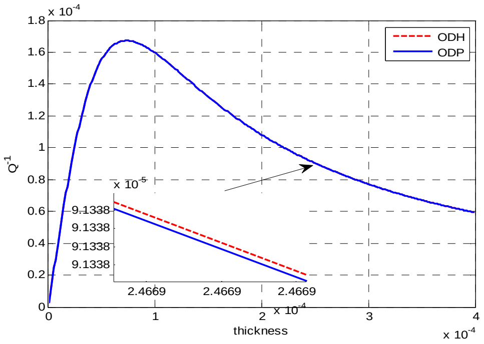

Fig. 3
Comparison of Q^{-1} for Parabolic and Hyperbolic heat conduction model for *R*/*h*=50 and $T_\circ = 300$.

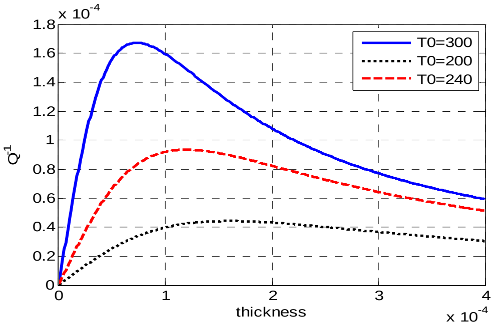

Fig. 4
Effect of the plate thickness on Q^{-1} for different ambient temperatures (R/h=50).

Fig. 2 shows variation of the quality factor versus different environment temperatures. As illustrated with increasing the number of used shape functions there is a good convergence and also a good agreement with the results of reference [4]. In Fig. 3 the results of thermoelastic damping obtained from the parabolic and hyperbolic heat conduction models are illustrated. As shown, the difference between two models in the micro scale is not significant. In addition the value of thermoelastic damping for a specific thickness well-known as critical thickness (h_c) has a maximum value [1, 12, 24].

5.1 Effect of thickness and ambient temperature on quality factor

In Fig. 4 the thermoelastic damping versus the plate thickness are shown for different ambient temperatures. As shown, with increasing the ambient temperature the value of the TED and critical thickness is increased. Fig. 5 shows variation of the thermoelastic damping versus microplate thicknesses for three different values of the micro-plate radius when $V_{DC} = 0$. As indicated, for different values of R the value of Q^{-1} in h_c are the same and changing in dimension of the micro-plate in constant ambient temperature does not change the value of Q^{-1} in critical thicknesses.

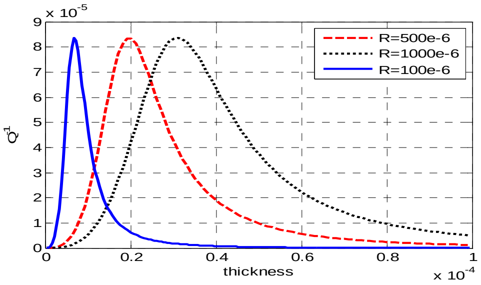

Fig. 5
Effect of plate radius on Q^{-1} for different thicknesses.

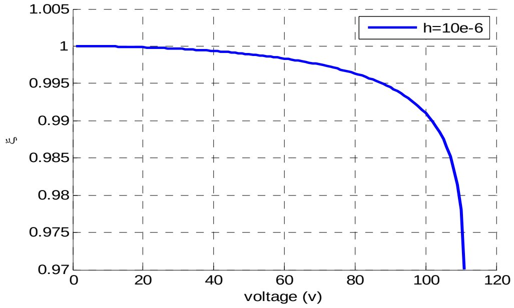

Fig. 6
Effect of DC voltage on ξ for $h=10\ \mu m$ and $E=250\ \mu m$.

5.2 Effect of DC voltage on quality factor

In this section, the calculated center gap of the microplate versus the applied voltage and the voltage leading the structure to an unstable condition through a saddle node bifurcation (pull-in voltage) is determined for $h = 10 \mu m$. Then the ratio of Quality factor of the micro-plate is calculated versus the applied voltage and shown in Fig. 6. The ratio of quality factor is defined as following:

$$\xi = \frac{Q}{Q_0} \tag{43}$$

where Q is the quality factor for different voltages and $Q_\circ$ is the quality factor when the applied voltage is zero $(V_{DC} = 0)$. This figure shows that the quality factor is decreased moderately with increasing the applied voltage but this decrease is very significant for the voltages near to the pull-in voltage.

6 CONCLUSION

The TED of an electro statically deflected circular micro plate in axi-symmetric out-of-plane vibrations was studied. The nonlinear equation of static deflection and small flexural free vibrations coupled with equation of temperature distribution were derived based on Kirchhoff classic thin plate Theory. The nonlinear equation of the static deflection was solved using Step by Linearization Method, and coupled linear small flexural thermo-elastic vibrations was solved implementing Galerkin based reduced order model. It was shown that the quality factor depends on ambient temperature and it is a significant loss mechanism at room temperature for micro-scale resonators. The results of parabolic and hyperbolic heat conduction models are compared and shown that the difference between these two cases is not significant in micro-plate resonators.

Thermoelastic damping and quality factor in circular micro-plate resonators under electrostatic loading can be affected by variation of the applied bias DC voltage. The results showed that the value of the thermoelastic damping is decreased by increasing the bias DC voltage. It is important to note that if the micro resonator is vibrating while is subjected to a bias DC voltage near to the pull-in voltage, energy dissipation would be out of control in this state, and any little increment in the bias DC voltage would considerably decrease the TED ratio. As the results indicated, therefore, it will be proper to get the bias DC voltage far from the pull-in voltage.

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