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Free Vibrations of Continuous Grading Fiber Orientation Beams on Variable Elastic Foundations

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ABSTRACT

Free vibration characteristics of continuous grading fiber orientation (CGFO) beams resting on variable Winkler and two-parameter elastic foundations have been studied. The beam is under different boundary conditions and assumed to have arbitrary variations of fiber orientation in the thickness direction. The governing differential equations for beam vibration are being solved using Generalized Differential Quadrature (GDQ) method. Numerical results are presented for a beam with arbitrary variation of fiber orientation in the beam thickness and compared with similar discrete laminate beam**.** The main contribution of this work is to present useful results for continuous grading of fiber orientation through thickness of a beam on variable elastic foundation and its comparison with similar discrete laminate composite beam. The results show the type of elastic foundation plays very important role on the natural frequency parameter of a CGFO beam. According to the numerical results, frequency characteristics of the CGFO beam resting on a constant Winkler elastic foundation is almost the same as of a composite beam with different fiber orientations for large values of Winkler elastic modulus, and fiber orientations has less effect on the natural frequency parameter. The interesting results show that normalized natural frequency of the CGFO beam is smaller than that of a similar discrete laminate beam and tends to the discrete laminated beam with increasing layers. It is believed that new results are presented for vibrational behavior of CGFO beams are of interest to the scientific and engineering community in the area of engineering design.

Keywords: continuous grading fiber orientation; free vibrations; beam; elastic foundation; GDQ method

1 INTRODUCTION

FUNCTIONALLY graded materials (FGMs) are heterogeneous materials in which the elastic and thermal properties change from one surface to the other, gradually and continuously. These structures support applied external forces efficiently by virtue of their geometrical shapes. Suresh and Mortensen [1] studied on functionally graded materials and describe the fundamentals of FGMs. Pradhan et al. [2] assumed that material properties follow a through-thickness variation according to a power-law distribution in terms of the volume fractions of constituents. FGM is a class of composite that has a smooth and continuous variation of material properties from one surface to another and thus can alleviate the stress concentrations found in laminated composites.

Beams and columns supported along their length are very common in structural configurations. Beam structures are often found to be resting on earth in various engineering applications. These include railway lines, geotechnical areas, highway pavement, building structures, offshore structures, transmission towers and transversely supported

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pipe lines. This motivated many researchers to analyze the behavior of beam structures on various elastic foundations. Studies on homogenous isotropic beams resting on variable Winkler foundation are found in various papers. Zhou Ding [3] presented a general solution for vibration of beam on variable Winkler elastic foundation. Employing the finite element method, Thambiratnam and Zhuge [4] studied the free vibration analysis of beams resting on elastic foundations. For two-parameter elastic foundation, Matsunaga [5] studied the linear vibration of non- prismatic beams resting on two-parameter elastic foundations. Ying et al. [6] presented solutions for bending and free vibration of FG beams resting on a Winkler–Pasternak elastic foundation based on the two-dimensional theory of elasticity. Yas et al. [7] studied free vibration analysis and optimization of functionally graded Euler–Bernoulli beams resting on two-parameter elastic foundations.

The differential quadrature method (DQM) is found to be a simple and efficient numerical technique for solving partial differential equations as reported by Bellman et al. [8]. The mathematical fundamental and recent developments of GDQ method as well as its major applications in engineering were discussed in detail in book [9]. Combination of the state-space method and the technique of DQ were used for free vibration of generally laminated beams by Chen et al. [10]. They did this by discretizing the state space formulations along the axial direction using the technique of DQ, new state equations at discrete pointes were established. Chen [11] used DQM to determine vibration characteristics of cross-ply laminated plates subjected to cylindrical bending. Khalili [12] used a mixed Ritz-DQ method to study the dynamic behavior of functionally graded beams subjected to moving loads and considered the material properties of the FG beam vary through the thickness according to exponential and power-law functions. Pradhan [13] studied thermo-mechanical vibration of FGM sandwich beam under variable elastic foundations by using GDQ method.

In this paper, we study the dynamic behavior of a new type of materials called "functionally graded fiber volume or fiber orientation materials". These kinds of materials have some advantages over discrete laminated ones. For these materials significant improvements are found in their applications due to the reduction in spatial mismatch of mechanical material properties. This research is in the continuation of our previous work [14] to consider vibrational behavior of CGFO beams on variable elastic foundation and its comparison with discrete laminated beam. The present work provides an enhanced insight into the mechanical behavior of this type of materials. For this purpose, a semi-analytic solution procedure for the free vibrations analysis of continuous grading fiber orientation beam on variable elastic foundation is presented. A detailed parametric study is carried out to highlight the influences of fiber orientation in the beam's thickness, material property graded indexes, coefficients of elastic foundations with constant, linear and parabolic modulus on the vibration frequencies of beams and finally comparison is made with similar discrete laminate composite beams.

2 PROBLEM DESCRIPTION

Consider a functionally graded fiber orientation beam with its coordinate system (x, z) as shown in Fig. 1. Three different Winkler elastic foundations including:

a: Winkler elastic foundation with constant modulus $k(x)=k_0$

b: Winkler elastic foundation with linear variation type $k(x)=k_0(1-\alpha x)$

c: Winkler elastic foundation with nonlinear variation type $k(x)=k_0(1-\beta x^2)$

are considered in this study. The beam is divided into N fictitious thin layers in the thickness direction, thus each layer can be considered as a plane stress state. The on-axis and off-axis coordinate systems coincide with the "1- 2" and "x-y" directions respectively. The mechanical constitutive relations, which relates the stresses to the strains for the Kth layer is expressed as [15]:

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix}^k = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & 0\\ \bar{Q}_{21} & \bar{Q}_{22} & 0\\ 0 & 0 & \bar{Q}_{66}\end{bmatrix}^k \begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix}^k \tag{1}$$

where Q_{ij} is the stiffness of the beam at the Kth layer.

The solution presented here is applicable for arbitrary variation of material composition through the thickness of the beam. For the beam, we assume the following specific power-law variation of the fiber orientation [14]:

$$\phi = \phi_i + (\phi_o - \phi_i)(\frac{1}{2} + \frac{z}{h})^p \tag{2}$$

where ϕ_i and ϕ_o denote the fiber orientation on the $z = -\frac{h}{2}$ and $z = \frac{h}{2}$ respectively and may typically ranges from 0^o to 90^o. The power p denotes the manner in which the orientation of the fibers varies through the thickness. Fig. 2 shows the variations of the fiber orientations through the thickness ($\eta = \frac{z}{h}$). In this figure the fiber orientations are assumed as $\phi_i = 0^o$ and $\phi_o = 90^o$ on the lower and upper surfaces respectively.

For continuous grading fiber orientation beam resting on variable two-parameter elastic foundation, the linear governing equation can be expressed as [7]:

$$-D_{fgm}\frac{\partial^4 w}{\partial x^4} + \frac{\partial}{\partial x}(k_1(x)\frac{\partial w}{\partial x}) - k(x)w - \rho A\frac{\partial^2 w}{\partial t^2} = 0 \qquad 0 \le x \le L \tag{3}$$

in which

$$D_{fgm} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \bar{Q}_{11}\ z^2\ dz \tag{4}$$

where $k(x), k_1(x)$ are Winkler and shearing layers elastic coefficients of the foundation. To obtain the natural frequency, the above equation is formulated as an eigen value problem by using the following periodic function:

$$w(x,t) = W(x)e^{-i\omega t} \tag{5}$$

where $W(x)$ is the mode shape of the transverse motion of the beam, therefore:

$$D_{fgm}\frac{\partial^4 W}{\partial x^4} - \frac{\partial}{\partial x}(k_1(x)\frac{\partial W}{\partial x}) + k(x)W + \rho A\omega^2 W = 0 \qquad 0 \le x \le L \tag{6}$$

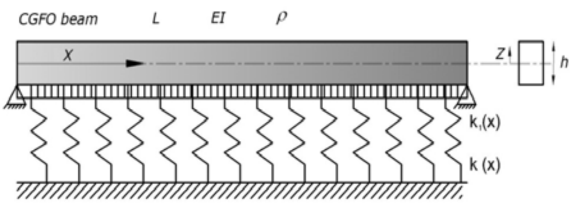
Eq. (6) is a fourth-order ordinary differential equation. Thus, it requires four boundary conditions. The following two types of boundary conditions are considered.

Simply supported edge

$$W = 0, \frac{\partial^2 W}{\partial x^2} = 0 \quad at \quad x = 0\ or\ x = L \tag{7}$$

Clamped edge

$$W = 0, \frac{\partial W}{\partial x} = 0 \quad at \quad x = 0\ or\ x = L \tag{8}$$



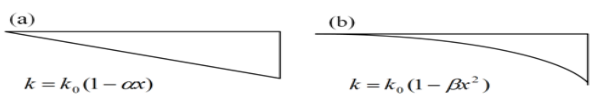


Fig.1
Various Winkler elastic foundations along the axial direction: (a) linear type, (b) parabolic type.

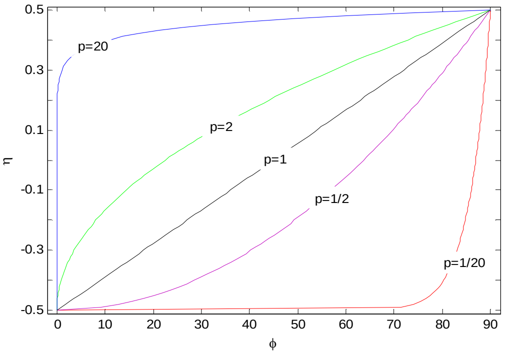


Fig.2
Variations of the fiber orientation (ϕ) through the thickness for different value of p.

3 GDQ SOLUTION OF GOVERNING EQUATION

The generalized differential quadrature (GDQ) approach is used to solve the governing equation of the beam. The GDQ approach was developed by Shu and coworkers [9, 16] based on the (DQ) technique [8]. It approximates the spatial derivative of a function of given grid point as a weighted linear sum of all the functional value at all grid point in the whole domain. The computation of weighting coefficient by GDQ is based on an analysis of a high order polynomial approximation and the analysis of a linear vector space. The weighting coefficients of the first-order derivative are calculated by a simple algebraic formulation, and the weighting coefficient of the second-and higher-order derivatives are given by a recurrence relationship. The details of the GDQ method can be found in [9, 16]. In the GDQ method, the *n*th order of a continuous function $f(x,z)$ with respect to x at a given point x_i can be approximated as a linear sum of weighting values at all of the discrete point in the domain of x , i.e. [9]:

$$\frac{\partial f^{n}(x_i,z)}{\partial x^{n}} = \sum_{k=1}^{N} c_{ik}^{n} f(x_{ik},z) \,, \quad (i=1,\, 2,\, N \;\;,\;\; n=1,\, 2,\, \ldots, n-1\,) \tag{9}$$

where N is the number of sampling points, and c_{ij}^{n} is the x_i dependent weight coefficients.

In order to determine the weighting coefficients c_{ij}^{n} , the Lagrange interpolation basic functions are used as test functions, and explicit formulation for computing these weighting coefficients can be obtained [9,16] :

$$c_{i,j}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)} \qquad , \quad i,j = 1,2,...,N, i \neq j \tag{10}$$

where

$$M^{(1)}(x_i) = \prod_{j=1, i\neq j}^{N} (x_i - x_j) \tag{11}$$

For the first-order derivative; i.e. n=1 and for higher-order derivative, one can use the following relations iteratively:

$$c_{i,j}^{n} = n\left(c_{i,i}^{(n-1)} c_{i,j}^{(1)} - \frac{c_{i,j}^{(n-1)}}{(x_i - x_j)} \right) \;, \quad i, j = 1,2,..., N, \quad i \neq j \;,\; n = 2,\, 3,\, n-1 \tag{12}$$

$$c_{i,i}^{(n)} = -\sum_{j=1, i\neq j}^{N} c_{i,j}^{(n)} \;, \quad i = 1,2,..., N, \quad n = 1,2,..., N-1 \tag{13}$$

A simple and natural choice of the grid distribution is the uniform grid spacing rule. However, it was found that non-uniform grid spacing yields results with better accuracy [17]. Hence, in this work, the Chebyshev-Gauss-Labatto quadrature points are used, that is [9]

$$x_i = \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{n-1}\pi\right)\right) \qquad i = 1,2,...,N \tag{14}$$

4 RESULTS AND DISCUSSION

First, verification study of the results is considered for an isotropic beam resting on Winkler elastic foundation in Table 1. As observed there is good agreement between the present results with similar ones obtained by Zhou Ding [3].

In this section, we characterize the response of the CGFO beam resting on different types of two-parameter elastic foundation. It is assumed the beam has the following mechanical properties [15]:

$$\frac{E_L}{E_T} = 25, \; \frac{G_{LT}}{E_T} = 0.2, \; \frac{G_{TT}}{E_T} = 0.5, \; \nu = 0.25$$

First the convergence of the method is investigated in evaluating the natural frequency parameter $\Omega = \omega L^2 \sqrt{\frac{\rho A}{E_T I}}$. The results are prepared for a graded beam with a linear variation of fiber orientation (*p=1*) from ϕ_i = 0° at $z = -\frac{h}{2}$ to ϕ_0 = 90° at $z = \frac{h}{2}$ and is shown in Fig. 3. Fast rate of convergence of the method is evident at different boundary conditions and it is found that only ten DQ grid in the thickness direction can yield accurate results. It is also observed for the considered system the formulation is stable while increasing the number of points and that the use of 50 points guarantees convergence of the procedure.

Now we compare a continuous grading fiber orientation beam with a linear variations of fiber orientation (*p =1*), from ϕ_i = 0° at $z = -\frac{h}{2}$ to ϕ_o = 90° at $z = \frac{h}{2}$ with discretely laminated 2-layer [0° /90°], 3-layer [0° /45° /90 °], 4-layer [0° /30° /60 ° /90 °] and 7-layer [0° /15° /30° /…/90°] respectively. This comparison is shown in Table 2. for various values of the wave numbers. It results the natural frequency of the CGFO beam is smaller that of a discrete laminate composite one. Also it is found that by increasing the layers of a discrete laminate composite beam, its natural frequency decreases and tends to a similar functionally graded fiber orientation one. Here we consider the effect of various Winkler elastic foundations on the CGFO beam with simply supported ends. Fig. 4 shows variations of the natural frequency parameter of a functionally graded fiber orientation as well as composite beams with different fiber orientations versus various constant Winkler elastic foundations. In this figure the shearing layers elastic coefficient (k_1) is assumed to be unity while Winkler elastic modulus (k) is considered to vary from 10 to 100,000. From this figure one could observe that for k>10000, the natural frequency parameter of the CGFO beam as well as composite one with different fiber orientation are the same. In other words for the large values of Winkler elastic modulus (k), fiber orientations has less effect on the natural frequency parameter. Fig. 5 shows variations of the natural frequency parameter of continuous grading fiber orientation beam with a linear variation of fiber orientation (*p=1*) from ϕ_i = 0° at $z = -\frac{h}{2}$ to ϕ_0 = 90° at $z = \frac{h}{2}$ resting on different types of Winkler elastic foundation. As observed, different types of Winkler elastic foundation doesn't affect on the natural frequency parameter of CGFO beam for Winkler elastic constant (k_0), ranges 10< k_0<1000 and then for k_0>1000, the natural frequency parameter of a CGFO beam resting on a variable Winkler elastic foundation decreases from constant type to parabolic and then linear types. In this figure the ends of the CGFO beam is simply supported. The effect of Winkler elastic foundation coefficients with constant modulus on the natural frequency parameter of a CGFO beam with simply supported ends is illustrated in Fig. 6 for different shearing layer coefficient. As it could be observed the natural frequency parameter converges with increasing the shearing layer elastic coefficient. For further study, the first three natural frequency parameters of the CGFO beam on Winkler elastic foundation with various linear modulus (α) as well as parabolic modulus (β) is shown for different boundary conditions in Tables 3. and 4. As noticed the natural frequencies parameter decrease with the increase of linear and parabolic modulus.

Table 1
Comparison of the frequency parameters of an isotropic beam resting on parabolic type of Winkler elastic foundation $(k = k_0(1-\beta x^2), k_1 = 0, \lambda_i^4 = \frac{\rho A L^4}{EI_0}\omega_i^2)$

k_0	β		λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
1000	0.4	Zhou[3]	5.597	7.022	9.675	12.675	15.763	18.882
		Present	5.5961	7.0231	9.6744	12.6743	15.7636	18.8818
	0.8	Zhou[3]	5.409	6.935	9.638	12.659	15.753	18.878
		Present	5.4100	6.9347	9.6385	12.6582	15.7552	18.8769
1500	0.4	Zhou[3]	6.138	7.321	9.792	12.728	15.792	18.898
		Present	6.1382	7.3207	9.7923	12.7273	15.7912	18.8979
	0.8	Zhou[3]	5.917	7.206	9.741	12.704	15.778	18.892
		Present	5.9162	7.2070	9.7407	12.7034	15.7786	18.8906
2000	0.4	Zhou[3]	6.564	7.587	9.905	12.780	15.819	18.913
		Present	6.5642	7.5865	9.9063	12.7797	15.8187	18.9140
	0.8	Zhou[3]	6.312	7.454	9.841	12.749	15.802	18.903

Table 2
Comparison of natural frequency parameter of CGFO beam with discretely laminated beam $(k_1 = 1, k = 2000)$

M	[0°/90°]	[0°/45°/90°]	[0°/30°/60°/90°]	[0°/15°/30°/…/90°]	CGFO(0°,90°)
1	57.9724	57.7744	57.4724	57.2721	56.9237
2	153.800	152.603	150.767	149.543	147.399
3	333.937	331.146	326.861	324.002	318.989
4	589.911	584.917	577.250	572.132	563.160
5	920.096	912.278	900.277	892.265	878.219
6	1324.07	1312.80	1295.51	1283.96	1263.72
7	1801.68	1786.35	1762.80	1747.08	1719.52
8	2352.88	2332.84	2302.08	2281.55	2245.55
9	2977.62	2952.26	2913.33	2887.34	2841.78
10	3675.90	3644.59	3596.53	3564.44	3508.18

Table 3
Variation of the first three natural frequency parameters of CGFO beam resting on the elastic foundation with various linear modulus *(k_1=1, $k=k_0(1-\alpha x)$)*

k_0	α	s-s			c-c			c-s		
		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
500	0	41.717	142.220	316.629	82.676	220.434	430.399	59.282	179.102	371.308
	0.2	41.113	142.044	316.550	82.373	220.320	430.341	58.800	178.957	371.239
	0.4	40.500	141.868	316.471	82.068	220.207	430.283	58.314	178.812	371.170
	0.6	39.877	141.692	316.392	81.763	220.093	430.225	57.825	178.666	371.102
1000	0	47.332	143.967	317.418	85.646	221.565	430.980	63.359	180.493	371.980
	0.2	46.263	143.619	317.260	85.060	221.339	430.864	62.455	180.204	371.843
	0.4	45.167	143.271	317.103	84.470	221.113	430.748	61.537	179.916	371.706
	0.6	44.042	142.923	316.945	83.875	220.887	430.632	60.604	179.627	371.569
2000	0	56.924	147.399	318.989	91.298	223.810	432.138	70.812	183.242	373.322
	0.2	55.136	146.720	318.676	90.195	223.363	431.907	69.186	182.674	373.049
	0.4	53.284	146.039	318.362	89.078	222.915	431.675	67.520	182.104	372.776
	0.6	51.359	145.356	318.048	87.946	222.467	431.444	65.809	181.534	372.503

Table 4
Variation of the first three natural frequency parameters of CGFO beam resting on the elastic foundation with various linear modulus *(k_1=1, $k=k_0(1-\beta x)$)*

k_0	β	S-S			C-C			C-S		
		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
500	0	41.717	142.220	316.629	82.676	220.434	430.399	59.282	179.102	371.308
	0.2	41.377	142.107	316.578	82.511	220.365	430.362	58.986	179.010	371.263
	0.4	41.033	141.994	316.526	82.346	220.296	430.326	58.689	178.918	371.219
	0.6	40.686	141.882	316.474	82.180	220.227	430.289	58.390	178.826	371.174
1000	0	47.332	143.967	317.418	85.646	221.565	430.980	63.359	180.493	371.980
	0.2	46.730	143.744	317.315	85.328	221.428	430.906	62.804	180.310	371.891
	0.4	46.119	143.521	317.211	85.008	221.291	430.833	62.245	180.127	371.802
	0.6	45.498	143.299	317.108	84.687	221.154	430.759	61.679	179.944	371.714
2000	0	56.924	147.399	318.989	91.298	223.810	432.138	70.812	183.242	373.322
	0.2	55.919	146.964	318.784	90.700	223.539	431.992	69.817	182.881	373.145
	0.4	54.891	146.529	318.578	90.097	223.268	431.845	68.805	182.521	372.968
	0.6	53.838	146.094	318.373	89.489	222.997	431.699	67.775	182.160	372.790

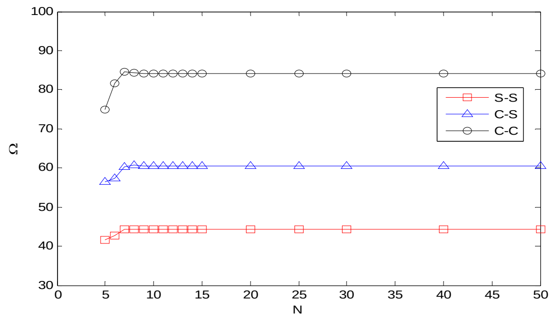


Fig.3
Convergency of the normalized natural frequency.

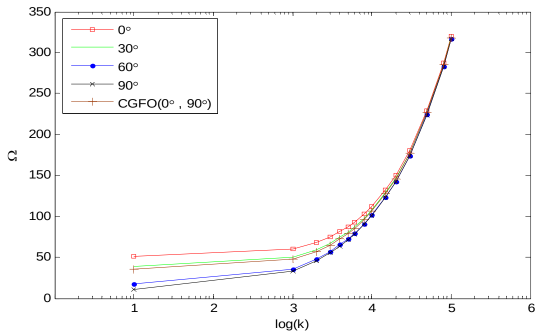


Fig.4
Effect of Winkler elastic foundation coefficients on the natural frequency parameter of CGFO as well as composite laminated beams *(k_1=1)*.

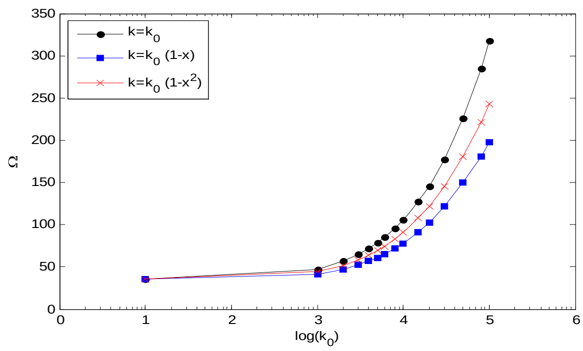


Fig.5
Variations of the natural frequency parameter of a CGFO beam resting on different kinds of Winkler elastic foundation $(k_1 = 1)$.

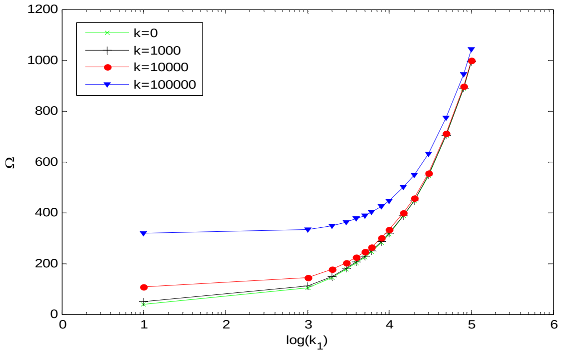


Fig.6
Variations of the natural frequency parameter vs. shearing layer elastic coefficient for different Winkler elastic foundation with constant modulus ($p=1$).

5 CONCLUSIONS

In this research work, the GDQ method has been used to study free vibration analysis of continuous grading fiber orientation (CGFO) beam. We checked the effectiveness of this method in predicting free vibration behavior of a functionally graded fiber orientation beams by comparing its results for isotropic condition with corresponding numerical results in the literature. From this study, some conclusions can be made:

- It has been found that the convergence of the GDQ results is very fast. The numerical results obtained by using only ten grid points agree very well with those in the literature.
- It results frequency characteristics of the CGFO beam behave very much the same as that of discrete laminate one. The new and interesting results show that the natural frequency of the CGFO beam is smaller that of a discrete laminate composite one and tends to the discrete laminated beam with increasing layers.
- It results frequency characteristics of the CGFO beam resting on a constant Winkler elastic foundation is almost the same as of a composite beam with different fiber orientations for large values of Winkler elastic modulus (k), and fiber orientations has less effect on the natural frequency parameter.
- The kind of Winkler elastic foundation doesn't affect on the natural frequency parameter of CGFO beam for Winkler elastic constant (k_0), ranges $10< k_0<1000$.
- It has been resulted the natural frequency parameter converges with increasing the shearing layer elastic coefficient
- It is noticed, the natural frequency parameter of a CGFO beam resting on a variable Winkler elastic foundation decreases from constant type to parabolic and then linear types.

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