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Effect of Magnetic Field on Torsional Waves in Non-Homogeneous Aeolotropic Tube

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ABSTRACT

The effect of magnetic field on torsional waves propagating in non-homogeneous viscoelastic cylindrically aeolotropic material is discussed. The elastic constants and non-homogeneity in viscoelastic medium in terms of density and elastic constant is taken. The frequency equations have been derived in the form of a determinant involving Bessel functions. Dispersion equation in each case has been derived and the graphs have been plotted showing the effect of variation of elastic constants and the presence of magnetic field. The obtained dispersion equations are in agreement with the classical result. The numerical calculations have been presented graphically by using MATLAB.

Keywords: Aeolotropic Material; Magnetic Field; Viscoelastic Solids; Non-Homogeneous; Bessel Functions

1 INTRODUCTION

THE theory of magneto elasticity describes the coupling of electromagnetic field and strain field, when a material is subjected to the electromagnetic field and then is deformed, as a result of which the electromagnetic field is changing and the strain field is arising. The interaction of elastic and electromagnetic fields has numerous applications in various field of science such as detection of mechanical explosions in the interior of the earth.

Very few problems of cylindrically aeolotropic elastic material have been considered as far because of the inherent difficulty in solving complicated simultaneous partial differential equations. Although much data is available on the propagation of surface waves but the torsional wave has not drawn much attention and very little literature is available on propagation of this wave. Kaliski and Petykiewicz [1], Narain [2] and many others have investigated the magneto elastic problems. However ,White and Tongtaow [3] has investigated cylindrical waves in transversely isotropic media. The problems of elastic wave propagation in the presence of a steady magnetic field have investigated by Das and Bhattacharya [4], Andreou and Dassios [5] and Suhubi [6]. Some of the analogous results on magneto elastic wave propagation problems, but in an anisotropic medium, were obtained by Abd-alla [7], Datta [8]. Acharya et al.[9] investigated the effect of the transverse isotropy and magnetic field on the interface waves in a conducting medium subject to the initial state of stress of the form of hydrostatic tension or compression. Liu and Chang [10] Studied the interactive behaviors among transverse magnetic fields, axial loads and external force of a magneto-elastic beam with general boundary conditions. Dai and Wang [11] Solved the magneto elastic

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wave propagation in hollow cylinder with arbitrary thickness. Tang and Xu [12] employed the method of Eigen function expansion to solve the problems of transient torsional vibration. Selim [13] investigated the effect of damping on the propagation of torsional waves in an initially stressed, dissipative, incompressible cylinder of infinite length. Chattopadhyay et al. [14] investigated the propagation of horizontally polarized shear waves in an internal magneto elastic monoclinic stratum with irregularity in lower interface.

In this study, an attempt has been made to investigate the torsional wave propagation in non-homogeneous viscoelastic cylindrically aeolotropic material permeated by a magnetic field. The graphs have been plotted showing the effect of variation of elastic constants and the presence of magnetic field. It is observed that the torsional elastic waves in a viscoelastic solid body propagating under the influence of a superimposed magnetic field can be different significantly from that of those propagating in the absence of a magnetic field.

2 BASIC EQUATIONS

The problem is dealing with magneto elasticity. Therefore, the basic equations will be electromagnetism and elasticity. The Maxwell equations of the electromagnetic field in a region with no charges ($\rho = 0$) and no currents ($J = 0$), such as in a vacuum, are [16]

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{1a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{1b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\overrightarrow{\partial B}}{\partial t}, \tag{1c}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\overrightarrow{\partial E}}{\partial t}. \tag{1d}$$

where, $\vec{E}$, $\vec{B}$, μ_0 and ε_0 are electric field, magnetic field induction, permeability and permittivity of the vacuum. For vacuum, $\mu_0 = 4\pi \times 10^{-7}$ and $\varepsilon_0 = 8.85 \times 10^{-12}$ in SI units. These equations lead directly to $\vec{E}$ and $\vec{B}$ satisfying the wave equation for which the solutions are linear combinations of plane waves traveling at the speed of light, $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$. In addition, $\vec{E}$ and $\vec{B}$ are mutually perpendicular to each other to the direction of wave propagation.

Also, the term Ohm's law is used to refer to various generalizations. The simplest example of this is:

$$\vec{J} = \sigma \vec{E}, \tag{2a}$$

where, $\vec{J}$ is the current density at a given location in a resistive material , $\vec{E}$ is the electric field at that location, and σ is a material dependent parameter called the conductivity. If an external magnetic field induction $\vec{B}$ is present and the conductor is not at rest but moving at velocity $\vec{V}$, then an extra term must be added to account for the current induced by the Lorentz force on the charge carriers [16].

$$\vec{J} = \sigma(\vec{E} + \vec{V} \times \vec{B}) = \sigma(\vec{E} + \frac{\partial v}{\partial t} \times \vec{B}). \tag{2b}$$

The electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a vacuum. The homogeneous form of the equation, written in terms of either the electric field $\vec{E}$ or the magnetic field induction $\vec{B}$, takes the form: [16]

$$\left(\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right)\vec{\mathrm{E}} = 0, \tag{3a}$$

$$\left(\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right)\vec{\mathrm{B}} = 0. \tag{3b}$$

where, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

The dynamical equations of motion in cylindrical coordinate (r,θ,z) are [15]

$$\frac{\partial s_{rr}}{\partial r} + \frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta} + \frac{\partial s_{rz}}{\partial z} + \frac{1}{r}(s_{rr} - s_{\theta\theta}) + T_R = \rho\frac{\partial^2 u}{\partial t^2}, \tag{4a}$$

$$\frac{\partial s_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta} + \frac{\partial s_{\theta z}}{\partial z} + \frac{2s_{r\theta}}{r} + T_\theta = \rho\frac{\partial^2 v}{\partial t^2}, \tag{4b}$$

$$\frac{\partial s_{rz}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta} + \frac{\partial s_{zz}}{\partial z} + \frac{s_{rz}}{r} + T_Z = \rho\frac{\partial^2 w}{\partial t^2}. \tag{4c}$$

where, $s_{rr}, s_{r\theta}, s_{rz}, s_{rr}.s_{\theta\theta}, s_{\theta z}, s_{zz}$ are the respective stress components, T_R, T_θ, T_Z are the respective body forces and u, v, w are the respective displacement components.

The stress-strain relations are [20]

$$s_{rr} = \delta_{11}^0 e_{rr} + \delta_{12}^0 e_{\theta\theta} + \delta_{13}^0 e_{zz}, \tag{5a}$$

$$s_{\theta\theta} = \delta_{21}^0 e_{rr} + \delta_{22}^0 e_{\theta\theta} + \delta_{23}^0 e_{zz}, \tag{5b}$$

$$s_{zz} = \delta_{31}^0 e_{rr} + \delta_{32}^0 e_{\theta\theta} + \delta_{33}^0 e_{zz}, \tag{5c}$$

$$s_{rz} = \delta_{44}^0 e_{rz}, \tag{5d}$$

$$s_{\theta z} = \delta_{55}^0 e_{\theta z}, \tag{5e}$$

$$s_{r\theta} = \delta_{66}^0 e_{r\theta}. \tag{5f}$$

where, $\delta_{ij} =$ elastic constants ($ij = 1, 2......6$).

The elastic constants of viscoelastic medium are [23]

$$\delta_{ij}^0 = \delta_{ij} + \delta_{ij}^{/}\frac{\partial}{\partial t} + \delta_{ij}^{//}\frac{\partial^2}{\partial t^2}(ij = 1, 2......6). \tag{6}$$

where, $\delta_{ij}^{/}$ and $\delta_{ij}^{//}$ are the first and second order derivatives of δ_{ij}.

The strain components are [22]

$$e_{rr} = \frac{1}{2}\frac{\partial u}{\partial r}, \tag{7a}$$

$$e_{\theta\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right), \tag{7b}$$

$$e_{zz} = \frac{1}{2}\frac{\partial w}{\partial z}, \tag{7c}$$

$$e_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial v}{\partial z}\right), \tag{7d}$$

$$e_{rz} = \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right), \tag{7e}$$

$$e_{zz} = \frac{1}{2}\frac{\partial w}{\partial z}, \tag{7f}$$

The rotational components are [22]

$$\Omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right), \tag{8a}$$

$$\Omega_\theta = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right), \tag{8b}$$

$$\Omega_z = \frac{1}{r}\left(\frac{\partial (rv)}{\partial r} - \frac{\partial u}{\partial \theta}\right). \tag{8c}$$

Equations governing the propagation of small elastic disturbances in a perfectly conducting viscoelastic solid having electromagnetic force $\left(\vec{J} \times \vec{\mathrm{B}}\right)$ (the Lorentz force, $\vec{J}$ is the current density and $\vec{\mathrm{B}}$ being magnetic induction vector) as the only body force are (using Eq. (4))

$$\frac{\partial s_{rr}}{\partial r} + \frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta} + \frac{\partial s_{rz}}{\partial z} + \frac{1}{r}(s_{rr} - s_{\theta\theta}) + \left(\vec{J} \times \vec{\mathrm{B}}\right)_R = \rho\frac{\partial^2 u}{\partial t^2}, \tag{9a}$$

$$\frac{\partial s_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta} + \frac{\partial s_{\theta z}}{\partial z} + \frac{2s_{r\theta}}{r} + \left(\vec{J} \times \vec{\mathrm{B}}\right)_\theta = \rho\frac{\partial^2 v}{\partial t^2}, \tag{9b}$$

$$\frac{\partial s_{rz}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta} + \frac{\partial s_{zz}}{\partial z} + \frac{s_{rz}}{r} + \left(\vec{J} \times \vec{\mathrm{B}}\right)_Z = \rho\frac{\partial^2 w}{\partial t^2}. \tag{9c}$$

Let us assume the components of magnetic field intensity $\vec{\mathrm{H}}$ are $\mathrm{H}_r = \mathrm{H}_\theta = 0$ and $\mathrm{H}_z = \mathrm{H}$ constant. Therefore, the value of magnetic field intensity is [9]

$$\vec{\mathrm{H}}\left(0,0,\mathrm{H}\right) = \vec{\mathrm{H}}_0 + \vec{\mathrm{H}}_i \tag{10}$$

where, $\vec{\mathrm{H}}_0$ is the initial magnetic field intensity along z-axis and $\vec{\mathrm{H}}_i$ is the perturbation in the magnetic field intensity.

The relation between magnetic field intensity $\vec{\mathrm{H}}$ and magnetic field induction $\vec{\mathrm{B}}$ is:

$$\vec{\mathrm{B}} = \mu_0 \vec{\mathrm{H}} \text{ (For vacuum, } \mu_0 = 4\pi \times 10^{-7} \text{ SI units.)} \tag{11}$$

From Eq. (1), Eq. (2), Eq. (3) and Eq. (10), we get,

$$\nabla^2 \vec{\mathrm{H}} = \mu_0 \sigma \left\{ \frac{\partial \vec{\mathrm{H}}}{\partial t} + \vec{\nabla} \times \left(\frac{\partial v}{\partial t} \times \vec{\mathrm{H}} \right) \right\} \tag{12}$$

The components of Eq. (12) can be written as [9]

$$\frac{\partial \mathrm{H}_r}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathrm{H}_{r,} \tag{13a}$$

$$\frac{\partial \mathrm{H}_\theta}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathrm{H}_{\theta,} \tag{13b}$$

$$\frac{\partial \mathrm{H}_z}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathrm{H}. \tag{13c}$$

3 FORMULATION OF THE PROBLEM

Let us consider a semi-infinite hollow cylindrical tube of radii α and β. The elastic properties of the shell are symmetrical about z-axis, and the tube is placed in an axial magnetic field surrounded by vacuum. Since, we are investigating the torsional waves in an aeolotropic cylindrical tube therefore the displacement vector has only v component. Hence,

$$u = 0, \tag{14a}$$

$$w = 0 \tag{14b}$$

$$v = v(r, z). \tag{14c}$$

Therefore, from Eq. (14) and Eq. (7), we get,

$$e_{rr} = e_{\theta\theta} = e_{zz} = e_{zr} = 0, \tag{15a}$$

$$e_{\theta z} = \frac{1}{2} \left(\frac{\partial v}{\partial z} \right), \tag{15b}$$

$$e_{r\theta} = \frac{1}{2} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right). \tag{15c}$$

From Eq. (14) and Eq. (8), we get,

$$\Omega_r = \frac{1}{2}\left(\frac{\partial v}{\partial z}\right), \tag{16a}$$

$$\Omega_\theta = 0, \tag{16b}$$

$$\Omega_z = \frac{\partial v}{\partial r} + \frac{v}{r}. \tag{16c}$$

Using Eq. (14), Eq. (15) and Eq. (6), the Eq. (5) becomes

$$s_{rr} = s_{\theta\theta} = s_{zz} = s_{rz} = 0, \tag{17a}$$

$$s_{r\theta} = (\delta_{66} + \delta'_{66}\frac{\partial}{\partial t} + \delta''_{66}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}), \tag{17b}$$

$$s_{\theta z} = (\delta_{55} + \delta'_{55}\frac{\partial}{\partial t} + \delta''_{55}\frac{\partial^2}{\partial t^2})(-\frac{1}{2}\frac{\partial v}{\partial r}). \tag{17c}$$

where, δ'_{ij} and δ''_{ij} are the first and second order derivatives of δ_{ij}.

For perfectly conducting medium, (i.e. $\sigma \to \infty$), it can be seen that Eq. (2) becomes

$$\vec{\mathrm{E}} = \left[-\frac{\mu_0 \mathrm{H}}{c}\frac{\partial v}{\partial t}, 0, 0\right] \tag{18}$$

Eq. (1) and Eq. (18), the Eq. (13) becomes,

$$\vec{\mathrm{H}}_i = \left[0, \mathrm{H}\frac{\partial v}{\partial z}, 0\right] \tag{19}$$

From the above discussion, the electric and magnetic components in the problem are related as:

$$\left[-\frac{\mu_0 \mathrm{H}}{c}\frac{\partial v}{\partial t}, 0, 0\right] = \left[0, \mathrm{H}\frac{\partial v}{\partial z}, 0\right] \tag{20}$$

Using Eq. (19) and Eq. (1) to get the components of body force in terms of Gaussian system of units as:

$$\mathrm{T} = \left[0, \frac{\mathrm{H}}{4\pi}\frac{\partial^2 v}{\partial z^2}, 0\right] \tag{21}$$

Eq. (17) and Eq. (20) satisfy the Eq. (4a) and Eq. (4c), therefore, the remaining Eq. (4b) becomes

$$\left\{\begin{array}{l} \frac{\partial}{\partial r}(\delta_{66} + \delta'_{66}\frac{\partial}{\partial t} + \delta''_{66}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}) + \frac{\partial}{\partial z}(\delta_{55} + \delta'_{55}\frac{\partial}{\partial t} + \delta''_{55}\frac{\partial^2}{\partial t^2})(-\frac{1}{2}\frac{\partial v}{\partial r}) \\ + \frac{2}{r}(\delta_{66} + \delta'_{66}\frac{\partial}{\partial t} + \delta''_{66}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}) - \frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2} \end{array}\right\} = \rho\frac{\partial^2 v}{\partial t^2} \tag{22}$$

Let

$$C_{ij} = \delta_{ij} r^{l}, C'_{ij} = \delta'_{ij} r^{l}, C''_{ij} = \delta''_{ij} r^{l} \qquad and \qquad \rho = \rho_0 r^{m} \tag{23}$$

where, δ_{ij}, δ'_{ij}, δ''_{ij} and ρ_0 are constants, r is the radius vector and l, m are non-homogeneities.

From Eq. (23), we get Eq. (17) as:

$$s_{r\theta} = (\delta_{66} + \delta'_{66}\frac{\partial}{\partial t} + \delta''_{66}\frac{\partial^2}{\partial t^2}) r^{l}\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}), \tag{24a}$$

$$s_{r\theta} = (\delta_{66} + \delta'_{66}\frac{\partial}{\partial t} + \delta''_{66}\frac{\partial^2}{\partial t^2}) r^{l}\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}), \tag{24b}$$

Using Eq. (23), the Eq. (22) becomes

$$\left\{\begin{array}{l} \frac{\partial}{\partial r}(\delta_{66} + \delta'_{66}\frac{\partial}{\partial t} + \delta''_{66}\frac{\partial^2}{\partial t^2}) r^{l}\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}) + \frac{\partial}{\partial z}(\delta_{55} + \delta'_{55}\frac{\partial}{\partial t} + \delta''_{55}\frac{\partial^2}{\partial t^2}) r^{l}(-\frac{1}{2}\frac{\partial v}{\partial r}) \\ +\frac{2}{r}(\delta_{66} + \delta'_{66}\frac{\partial}{\partial t} + \delta''_{66}\frac{\partial^2}{\partial t^2}) r^{l}\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}) - \frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2} \end{array}\right\} = \rho_0 r^{m}\frac{\partial^2 v}{\partial t^2} \tag{25}$$

4 SOLUTION OF THE PROBLEM

Let $v = \xi(r) e^{i(\varsigma z + \zeta t)}$ [18] be the solution of Eq. (25). Hence, Eq. (25) reduces to

$$\frac{\partial^2 \xi}{\partial r^2} + \frac{(l+1)}{r}\frac{\partial \xi}{\partial r} - \frac{(l+1)}{r^2}\xi + \Theta_1^2 \xi + \Theta_2^2 \frac{\xi}{r^{l}} = 0 \tag{26}$$

where

$$\Theta_1^2 = \frac{2\rho_0 \zeta^2 - (\delta_{55} + \delta'_{55} i\zeta - \delta''_{55}\zeta^2)\varsigma^2}{\delta_{66} + \delta'_{66} i\zeta - \delta''_{66}\zeta^2}, \tag{27a}$$

$$\Theta_2^2 = \frac{H^2 \varsigma^2}{2\pi(\delta_{66} + \delta'_{66} i\zeta - \delta''_{66}\zeta^2)}. \tag{27b}$$

Eq. (26) is in complex form, therefore we generalize its solution for $l = 0$ and $l = 2$

4.1 Solution for $l = 0$

For, $l = 0$ the Eq. (26) becomes,

$$\frac{\partial^2 \xi}{\partial r^2}+\frac{1}{r}\frac{\partial \xi}{\partial r}+(\Xi^2-\frac{1}{r^2})\xi=0 \tag{28}$$

where,

$$\Xi^2=\Theta_1^2+\Theta_2^2 \tag{29}$$

The solution of Eq. (28) is

$$v=\{PJ_1(Gr)+QX_1(Gr)\}e^{i(\varsigma z+\zeta t)} \tag{30}$$

From Eq. (24) and Eq. (30)

$$s_{r\theta}=\{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2\}\left[\frac{P}{2}\{GJ_0(Gr)-\frac{2}{r}J_1(Gr)+\frac{Q}{2}\{GX_0(Gr)-\frac{2}{r}X_1(Gr)\right]e^{i(\varsigma z+\zeta t)} \tag{31}$$

5 BOUNDARY CONDITIONS AND FREQUENCY EQUATION

The boundary conditions that must be satisfied are :

B1. For $r=\alpha$, (α is the internal radius of the tube)

$s_{r\theta}+\tau_{r\theta}=\tau_{(r\theta)_0}$

B2. For $r=\beta$, (β is the external radius of the tube)

$s_{r\theta}+\tau_{r\theta}=\tau_{(r\theta)_0}$

where $\tau_{r\theta}$ and $\tau_{(r\theta)_0}$ are the Maxwell stresses in the body and in the vacuum, respectively. There will be no impact of these Maxwell stresses. Hence,

$$\tau_{r\theta}=\tau_{(r\theta)_0}=0 \tag{32}$$

On simplification, Eq. (18) and Eq. (30) gives

$$\mathrm{E}=-\frac{\mu_0\mathrm{H}}{c}i\zeta\{PJ_1(Gr)+QX_1(Gr)\}e^{i(\varsigma z+\zeta t)} \tag{33}$$

Let , $\mathrm{E}_0=\Psi e^{i(\varsigma z+\zeta t)}$

Hence, Eq. (3) becomes

$$\frac{\partial^2 \Psi}{\partial r^2}+\frac{1}{r}\frac{\partial \Psi}{\partial r}+\gamma^2\Psi=0 \tag{34}$$

where

$$\gamma^2=\frac{\zeta^2}{c^2}-\varsigma^2 \tag{35}$$

The solution of the Eq. (34) becomes

$$\Psi = RJ_0(\gamma r) + SX_0(\gamma r) \tag{36}$$

where J_0 and X_0 are Bessel functions of order zero. R and S are constants.

From Eq. (37) and Eq. (40)

$$\Psi = \{RJ_0(\gamma r) + SX_0(\gamma r)\}e^{i(\varsigma z + \zeta t)} \tag{37}$$

The boundary conditions B1 and B2 with the help of the Eq. (31) and (32) turn into:

$$P\{G\alpha J_0(G\alpha) - 2J_1(G\alpha)\} + Q\{G\alpha X_0(G\alpha) - 2X_1(G\alpha)\} = 0 \tag{38}$$

$$P\{G\beta J_0(G\beta) - 2J_1(G\beta)\} + Q\{G\beta X_0(Ga) - 2X_1(G\beta)\} = 0 \tag{39}$$

Eliminating P and Q from Eq. (38) and Eq. (39)

$$\begin{vmatrix} G\alpha J_0(G\alpha) - 2J_1(G\alpha) & G\alpha X_0(G\alpha) - 2X_1(G\alpha) \\ G\beta J_0(G\beta) - 2J_1(G\beta) & G\beta X_0(Ga) - 2X_1(G\beta) \end{vmatrix} = 0 \tag{40}$$

On solving Eq. (40), we get the obtained frequency equation

$$\frac{G\alpha J_0(G\alpha) - 2J_1(G\alpha)}{G\beta J_0(G\beta) - 2J_1(G\beta)} = \frac{G\alpha X_0(G\alpha) - 2X_1(G\alpha)}{G\beta X_0(Ga) - 2X_1(G\beta)} = 0 \tag{41}$$

On the theory of Bessel functions, if tube under consideration is very thin i.e. $\beta = \alpha + \Delta\alpha$ and neglecting $\Delta\alpha^2, \Delta\alpha^3$........, the frequency equation can be written as [18]

$$\Xi^3\alpha^2 + \Xi - 1 = 0 \tag{42}$$

where

$$\Xi^2 = \frac{2\rho_0\zeta^2 - (\delta_{55} + \delta_{55}^{/} i\zeta - \delta_{55}^{//}\zeta^2)\varsigma^2 + \dfrac{H^2}{2\pi}\varsigma^2}{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2} \tag{43}$$

Putting the value of Ξ in Eq. (42), the frequency ζ of the wave can be found. Clearly, frequency ζ is dependent on magnetic field.

Put , $\Xi\alpha = \Phi$ (44)

The phase velocity $c_1 = \zeta / \varsigma$ can be written as:

$$\frac{c_1^2}{c_0^2} = \Phi^2\left(\frac{\lambda}{2\pi\alpha}\right)^2 + \mathrm{K} - \frac{\left(\dfrac{H^2}{4\pi}\right)}{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2} \tag{45}$$

where

$$\lambda = \frac{2\pi}{k}, \mathrm{K} = \frac{\delta_{55} + \delta'_{55} i\zeta - \delta''_{55}\zeta^2}{\delta_{66} + \delta'_{66} i\zeta - \delta''_{66}\zeta^2}, c_0^2 = \frac{\delta_{66} + \delta'_{66} i\zeta - \delta''_{66}\zeta^2}{2\rho_0} \tag{46}$$

The term H i.e. magnetic field is negative in Eq. (45) which reduces the phase velocity of torsional wave.

Case 1
Since the pipe under consideration is made of an aeolotropic material, then

$$\delta'_{ij} = \delta''_{ij} = 0 \tag{47}$$

Hence, from Eq. (42), Eq. (44) and Eq. (47) the frequency equation becomes

$$\Phi_0^3 + \Phi_0 - \alpha = 0 \tag{48}$$

Using Eq. (45) and Eq. (46), the phase velocity is

$$c_2^2 = \frac{\delta_{66}}{2\rho_0}\left\{\Phi_0^2\left(\frac{\lambda}{2\pi\alpha}\right)^2 + \frac{\delta_{55}}{\delta_{66}} - \frac{H^2}{2\pi\delta_{66}}\right\} \Rightarrow \frac{c_2}{c_0} = \left\{\frac{[\frac{\Phi_0}{2\pi}]^2}{[\frac{\alpha}{\lambda}]^2} + \frac{\delta_{55}}{\delta_{66}} - \frac{H^2}{2\pi\delta_{66}}\right\}^{\frac{1}{2}} \tag{49}$$

where

$$c_0^2 = \delta_{66}/2\rho_0 \tag{50}$$

The term H i.e. magnetic field is negative in Eq. (49) which reduces the phase velocity of torsional wave. This is in complete agreement with the corresponding classical results [17]

Case 2
If the pipe under consideration is made of an isotropic material, then

$$\delta'_{ij} = \delta''_{ij} = 0, \delta_{55} = \delta_{66} = \chi \tag{51}$$

Using Eq. (49) and Eq. (50), the phase velocity is

$$c_2^2 = \frac{\chi}{2\rho_0}\left\{\Phi_0^2\left(\frac{\lambda}{2\pi\alpha}\right)^2 + 1 - \frac{H^2}{2\pi\chi}\right\} \tag{52}$$

This is in complete agreement with the corresponding classical results [2]

5.1 Solution for l=2

For, $l = 2$ the Eq. (26) becomes,

$$\frac{\partial^2\xi}{\partial r^2} + \frac{3}{r}\frac{\partial\xi}{\partial r} + (\Theta_1^2 - \frac{(3-\Theta_2^2)}{r^2}\xi = 0 \tag{53}$$

Putting $\xi = \frac{1}{r}\mathrm{N}(r)$ in Eq. (53), one get

$$\frac{\partial^2 \mathrm{N}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{N}}{\partial r} + \left[\Theta_1^2 - \frac{\mathrm{P}^2}{r^2}\right]\mathrm{N} = 0 \tag{54}$$

where

$$\mathrm{P}^2 = 3 - \Theta_2^2 \tag{55}$$

Solution of Eq. (54) will be [18]

$$\mathrm{N} = RJ_{\mathrm{P}}(\Theta_1 r) + SX_{\mathrm{P}}(\Theta_2 r) \tag{56}$$

Putting the value of ξ and N in Eq. (55), we get

$$\mathrm{P} = \frac{1}{r}\{RJ_{\mathrm{P}}(\Theta_1 r) + SX_{\mathrm{P}}(\Theta_1 r)\}e^{i(\varsigma z + \zeta t)} \tag{57}$$

From the Eq. (24) and Eq. (57)

$$s_{r\theta} = (\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2)\begin{bmatrix} \frac{R}{2}\{\Theta_1 r J_{\mathrm{P}-1}(\Theta_1 r) - (\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1 r)\} \\ +\frac{S}{2}\{\Theta_1 r X_{\mathrm{P}-1}(\Theta_1 r) - (\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1 r)\} \end{bmatrix} e^{i(\varsigma z + \zeta t)} = 0 \tag{58}$$

with the help of Eq. (32), Eq. (57) and boundary conditions B1 and B2, we get

$$\frac{R}{2}\{\Theta_1\alpha J_{\mathrm{P}-1}(\Theta_1\alpha) - (\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\alpha)\} + \frac{S}{2}\{\Theta_1\alpha X_{\mathrm{P}-1}(\Theta_1\alpha) - (\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\alpha)\} = 0 \tag{59}$$

$$\frac{R}{2}\{\Theta_1\beta J_{\mathrm{P}-1}(\Theta_1\beta) - (\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\beta)\} + \frac{S}{2}\{\Theta_1\beta X_{\mathrm{P}-1}(\Theta_1\beta) - (\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\beta)\} = 0 \tag{60}$$

Eliminating R and S from Eq. (59) and Eq. (60)

$$\begin{vmatrix} \{\Theta_1\alpha J_{\mathrm{P}-1}(\Theta_1\alpha) - (\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\alpha)\} & \{\Theta_1\alpha X_{\mathrm{P}-1}(\Theta_1\alpha) - (\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\gamma\alpha)\} \\ \{\Theta_1\beta J_{\mathrm{P}-1}(\Theta_1\beta) - (\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\beta)\} & \{\Theta_1\beta X_{\mathrm{P}-1}(\Theta_1\beta) - (\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\gamma\beta)\} \end{vmatrix} = 0 \tag{61}$$

On solving Eq. (61), we get

$$\frac{\{\Theta_1\alpha J_{\mathrm{P}-1}(\Theta_1\alpha) - (\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\alpha)\}}{\{\Theta_1\alpha X_{\mathrm{P}-1}(\Theta_1\alpha) - (\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\alpha)\}} = \frac{\{\Theta_1\beta J_{\mathrm{P}-1}(\Theta_1\beta) - (\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\beta)\}}{\{\Theta_1\beta X_{\mathrm{P}-1}(\Theta_1\beta) - (\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\beta)\}} \tag{62}$$

If η_1 is the root of the above equation, then

$$\frac{\{\eta_1 J_{\mathrm{P}-1}(\eta_1) - (\mathrm{P}+2)J_{\mathrm{P}}(\eta_1)\}}{\{\eta_1 X_{\mathrm{P}-1}(\eta_1) - (\mathrm{P}+2)X_{\mathrm{P}}(\eta_1)\}} = \frac{\{\eta_1 F_1 J_{\mathrm{P}-1}(\eta_1 F_1) - (\mathrm{P}+2)J_{\mathrm{P}}(\eta_1 F_1)\}}{\{\eta_1 F_1 X_{\mathrm{P}-1}(\eta_1 F_1) - (\mathrm{P}+2)X_{\mathrm{P}}(\eta_1 F_1)\}} \tag{63}$$

where, $F_1 = \beta/\alpha$

On the theory of Bessel functions, if tube under consideration is very thin i.e. $\beta = \alpha + \Delta\alpha$ and neglecting $\Delta\alpha^2, \Delta\alpha^3$........, the frequency equation can be written as [18]

$$(\mathrm{P}+2)^2 - \left(2\mathrm{P} - 1 + \frac{1}{\Theta_1}\right)(\mathrm{P}+2) + \Theta_1^2\alpha^2 = 0 \tag{64}$$

where

$$\mathrm{P}^2 = 3 - \Theta_2^2 \Rightarrow \mathrm{P}^2 = 3 - \frac{H^2\varsigma^2}{2\pi(\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2)}, \tag{65a}$$

$$\Theta_1^2 = \frac{2\rho_0\zeta^2 - (\delta_{55} + \delta_{55}^{/} i\zeta - \delta_{55}^{//}\zeta^2)\varsigma^2}{\delta_{66} + \delta_{66}^{/} i\varsigma - \delta_{66}^{//}\varsigma^2}. \tag{65b}$$

From the Eq. (63), Eq. (64) and Eq. (65), the phase velocity can be written as (same as above Eq. (45) and Eq. (46))

$$\frac{c^2}{c_0^2} = \eta^2\left(\frac{\lambda}{2\pi\alpha}\right)^2 + \frac{\delta_{55} + \delta_{55}^{/} i\zeta - \delta_{55}^{//}\zeta^2}{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2} \tag{66}$$

Case 1

Since the pipe under consideration is made of an aeolotropic material, then

$$\delta_{ij}^{/} = \delta_{ij}^{//} = 0 \tag{67}$$

The frequency equation is given by

$$\frac{\{\Theta_3\alpha J_{\mathrm{P}_1-1}(\Theta_3\alpha) - (\mathrm{P}+2)J_{\mathrm{P}_1}(\Theta_3\alpha)\}}{\{\Theta_3\alpha X_{\mathrm{P}_1-1}(\Theta_3\alpha) - (\mathrm{P}+2)X_{\mathrm{P}_1}(\Theta_3\alpha)\}} = \frac{\{\Theta_3\beta J_{\mathrm{P}_1-1}(\Theta_3\beta) - (\mathrm{P}+2)J_{\mathrm{P}_1}(\Theta_3\beta)\}}{\{\Theta_3\beta X_{\mathrm{P}_1-1}(\Theta_3\beta) - (\mathrm{P}+2)X_{\mathrm{P}_1}(\Theta_3\beta)\}} \tag{68}$$

$$\eta_2^3 + 6\eta_2 - 3\alpha = 0 \tag{69}$$

$$\mathrm{P}_1^2 = 3 - \frac{H^2\varsigma^2}{2\pi\delta_{66}}, \qquad \Theta_3^2 = \frac{2\rho_0\zeta^2 - \delta_{55}\varsigma^2}{\delta_{66}}, \qquad \eta_2 = \Theta_3\varsigma \qquad \mathit{at}\ \mathrm{P}_1 = 1 \tag{70}$$

Using Eq. (66), Eq. (67), Eq. (68) and Eq. (70), we get (calculations are done in the similar manner as for the Eq. (48) to Eq. (50) for $l = 0$ case)

$$\frac{c_3}{c_{01}} = \left[\frac{\left(\dfrac{\eta_2}{2\pi}\right)^2}{\left(\dfrac{\alpha}{\lambda}\right)^2} + \frac{\delta_{55}}{\delta_{66}}\right]^{\frac{1}{2}} \tag{71}$$

where ${c_{01}}^2 = \delta_{66}/2\rho_0$

Case 2
If the pipe under consideration is made of an isotropic material, then

$$\delta_{ij}^{/} = \delta_{ij}^{//} = 0, \delta_{55} = \delta_{66} = \chi \tag{72}$$

The frequency equation (calculations are done as for the l=0 case) is

$$\frac{\{\Theta_4 \alpha J_{P_2-1}(\Theta_4\alpha) - (P+2) J_{P_2}(\Theta_4\alpha)\}}{\{\Theta_4 \alpha X_{P_2-1}(\Theta_4\alpha) - (P+2) X_{P_2}(\Theta_4\alpha)\}} = \frac{\{\Theta_4 \beta J_{P_2-1}(\Theta_4\beta) - (P+2) J_{P_2}(\Theta_4\beta)\}}{\{\Theta_4 \beta X_{P_2-1}(\Theta_4\beta) - (P+2) X_{P_2}(\Theta_4\beta)\}} \tag{73}$$

where

$$P_2^2 = 3 - \frac{H^2 \varsigma^2}{\chi}, \qquad \Theta_4^2 = \frac{2\rho_0 \zeta^2 - \chi \varsigma^2}{\chi}.$$

Using Eq. (72) and Eq. (73), the phase velocity for this case is (same as above Eq. (45) and Eq. (46))

$$\frac{c_4^2}{c_{02}^2} = \left[\frac{\left(\frac{\eta_2}{2\pi}\right)^2}{\left(\frac{\alpha}{\lambda}\right)^2} + 1 \right] \tag{74}$$

where, $c_{02}^2 = \chi / 2\rho_0$

6 NUMERICAL RESULTS

The effect of non-homogeneity on torsional waves in an aeolotropic material made of viscoelastic solids has been studied. The numerical computation of phase velocity has been made for homogeneous and non-homogeneous pipe. The graphs are plotted for the two cases (l=0 and l=2). Different values of α/λ (diameter/wavelength) for homogeneous in the presence of magnetic field and non homogeneous case in the absence of magnetic field are calculated from Eq. (49) and Eq. (66) with the help of MATLAB. The variations elastic constants and presence of magnetic field in two curves have been obtained by choosing the following parameters for homogeneous and non-homogeneous aeolotropic pipe.

$l = 0, \quad \Phi_0 = 2.333, \quad \alpha = 10, \quad \delta_{55} / \delta_{66} = 0.8, \quad H = .32G.$

$l = 2, \quad \Phi_0 = 2.333, \quad \alpha = 10, \quad \delta_{55} / \delta_{66} = 0.8, \quad H = 0$

The curves obtained in Fig. 1 clearly show that the phase velocity for homogeneous as well as non-homogeneous case decreases inside the aeolotropic tube. The presence of magnetic field also reduces the speed of tortional waves in viscoelastic solids. These curves justify the results obtained in Eq. (50) and Eq. (52) mathematically given by [2] and [17]. We see that for homogeneous case when magnetic field is present and for non-homogeneous case when magnetic field is not present the variation i.e. shape of the curves is the same. For non-homogeneous case, the elastic constants and the density of the tube are varying as the square of the radius vector.

Table 1
Material properties

	l	H (Gauss)	$\delta_{55} / \delta_{66}$
Homogeneous Pipe	0	0.32	0.8
Inhomogeneous Pipe	2	0	0.8

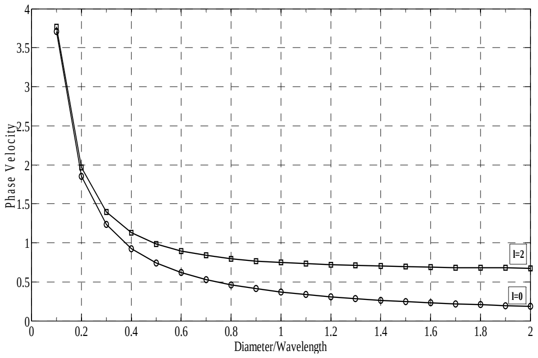

Fig. 1
Torsional wave dispersion curves.

7 CONCLUSIONS

The above problem deals with the interaction of elastic and electromagnetic fields in a viscoelastic media. This study is useful for detections of mechanical explosions inside the earth. In this study, an attempt has been made to investigate the torsional wave propagation in non-homogeneous viscoelastic cylindrically aeolotropic material permeated by a magnetic field. It has been observed that the phase velocity decreases as the magnetic field increases.

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