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Rheological Response and Validity of Viscoelastic Model Through Propagation of Harmonic Wave in Non-Homogeneous Viscoelastic Rods

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ABSTRACT

This study is concerned to check the validity and applicability of a five parameter viscoelastic model for harmonic wave propagating in the non-homogeneous viscoelastic rods of varying density. The constitutive relation for five parameter model is first developed and validity of these relations is checked. The non-homogeneous viscoelastic rods are assumed to be initially unstressed and at rest. In this study, it is assumed that density, rigidity and viscosity of the specimen i.e. rod are space dependent. The method of non-linear partial differential equation (Eikonal equation) has been used for finding the dispersion equation of harmonic waves in the rods. A method for treating reflection at the free end of the finite non-homogeneous viscoelastic rod is also presented. All the cases taken in this study are discussed numerically and graphically with MATLAB.

Keywords: Harmonic waves; Viscoelastic media; Friedlander series; Inhomogeneous; Varying density

1 INTRODUCTION

THE theory of viscoelasticity is useful in the field of solid mechanics, engineering, seismology, exploration and geophysics. Harmonic waves are waves that have vibrations perpendicular to their direction of travel i.e. waves in which the motion of the medium is in the perpendicular direction as the motion of the wave. The solutions of many problems related with wave-propagation for homogeneous media are available in many literatures of continuum mechanics of solids. In recent years, however, sufficient interest has arisen towards non-homogeneous bodies. Many researchers Alfrey [1], Barberan [2], Achenbach [3], Bhattacharya [4] and Acharya [5] formulated and developed the theory of elasticity. Further, Bert [6], Biot [7], Batra [8] successfully applied this theory to wave-propagation in homogeneous, elastic media. On the basis of the theory of elasticity, the propagation of harmonic waves in isotropic or anisotropic materials have been evaluated numerically by White [9], Mirsky [10] and Tsai [11]. Murayama [12] and Schiffman et al. [13] have proposed higher order viscoelastic models of five and seven parameters to represent the soil behavior. Gurdarshan [14], Kakar et al.; [15] and Kaur et al.; [16] analyzed five parameter and four parameter viscoelastic models for different waves and also studied these models under dynamic loadings. Recently, Kakar and Kaur [17, 18] have studied the responses of five parameter and four parameter viscoelastic models for different waves in non-homogeneous viscoelastic rods.

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In most of the literature, the problems of non-homogeneity are taken as independent of space coordinate. But in this study, we consider the wave propagation in non-homogeneous media, when density $'\rho'$ rigidity $'G'$ and viscosity $'\eta'$ of the material are space dependent such that the wave velocity is also space dependent. Here it is assumed that density, rigidity and viscosity of the specimen obey the law $\rho = \rho_0(1+\cos\alpha_1 x)$, $G = G_0(1+\cos\alpha_2 x)$ and $\eta = \eta_0(1+\cos\alpha_3 x)$ respectively. The problem is solved with Eikonal equation when the wave equation is approximated using WKB theory. The displacements assumed in the problem are so small that under isothermal conditions, the linear constitutive laws hold. The displacement and stress expressions are solved for time dependent displacement and stress boundary conditions. The paper ends with numerical analysis by taking material parameters.

2 THE CONSTITUTIVE RELATION FOR FIVE PARAMETER MODEL

We consider a five parameter model consists of two springs $S_1(G_1), S_2(G_2)$ where G_1 and G_2 are the modulli of elasticity associated to them and three dash-pots $D_2(\eta_2), D_{2'}(\eta_{2'}), D_3(\eta_3)$ where η_2, $\eta_{2'}$ and η_3 are the Newtonian viscosity coefficients associates to these elements. The module of elasticity and viscosity coefficients are assumed to be space dependent i.e. functions of $'x'$ in inhomogeneous case taken into consideration. Unidirectional problem is formed by taking the material in the form of filament of non-homogeneous viscoelastic material by taking one end at x = 0. The co-ordinate *x* is measured positive in the direction of the axis of the filament. Time is specified by *t*, and σ, γ and u respectively specify the only non-zero components of stress, shearing strain and particle displacement. The model has been divided into three sections, I, II, III. In Fig.1, the section I, section II and section III has one spring $S_1(G_1)$, two dash-pots $D_2(\eta_2), D_{2'}(\eta_{2'})$ one spring $S_2(G_2)$ and one dash-pot $D_3(\eta_3)$ respectively. [15, 17]

Under the supper- supposition principle, strains are added in the case of series connections and stresses are added when they are in parallel. Now if a_1, a_2, a_3 be the three shearing strains elongations in respective sections connected in series, then total elongation is $a = a_1 + a_2 + a_3$. The total stress in the network remains the same. In each section but, in the case of section II which is sub-divided into two added sections i.e. $\sigma = \sigma_1 + \sigma_2$, where σ_1 and σ_2 are the stresses in the sub-sections. Relation for stress and strain for $D_{2'}(\eta_{2'})$ for section II (represented by single dash-pot) is [23]

$$\sigma_1 = \eta_{2'} a_2 \tag{1}$$

Since the sub-section II is represented by a Maxwell- element, then the relation is expressed as:

$$\left(\frac{D}{G_2} + \frac{1}{\eta_2}\right)\sigma_2 = D(a_2) \tag{2}$$

Since, $\sigma = \sigma_1 + \sigma_2$ for Section II , therefore

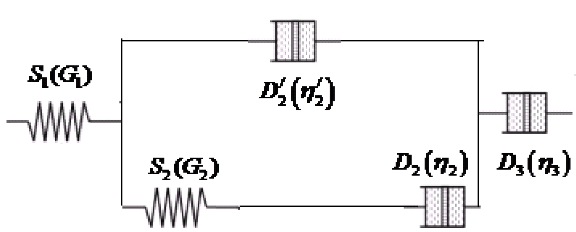

Fig. 1
Five parameter viscoelastic model.

$$\left(D+\frac{G_2}{\eta_2}\right)\sigma=\eta_{2'}\left[D^2+G_2\left(\frac{1}{\eta_{2'}}+\frac{1}{\eta_2}\right)D\right]a_2 \tag{3}$$

For section I, for the spring $S_1(G_1)$, the stress-strain relation is given by

$$\sigma=G_1a_1 \tag{4}$$

For Section III; for the dash-pot $D_3\left(\eta_3\right)$, the stress –strain relation is given by

$$\sigma=\eta_3\dot{a}_3 \tag{5}$$

The Stress-strain relation for the model representing the viscoelastic body for total stress (σ) and strain (a) can be obtained from Eq. (3), Eq. (4) and Eq. (5) as:

$$\left[D^2+\left\{\frac{G_1}{\eta_{2'}}+\frac{G_1}{\eta_3}+\frac{G_2}{\eta_{2'}}+\frac{G_2}{\eta_2}\right\}D+\left\{\frac{G_1}{\eta_{2'}}\frac{G_2}{\eta_2}+\frac{G_1}{\eta_3}\left(\frac{G_2}{\eta_{2'}}+\frac{G_2}{\eta_2}\right)\right\}\right]\sigma=G_1\left\{D^2+\left(\frac{G_2}{\eta_{2'}}+\frac{G_2}{\eta_2}\right)D\right\}a \tag{6}$$

Now we take

$$\tau_{ij}^{-1}=\theta_{ij}=\frac{G_i}{\eta_j}=\frac{S_i(G_i)}{Dj(\eta_j)} \tag{7}$$

where, $S_i(G_i)$ elastic modulus of spring and $Dj(\eta_j)$ = viscosity of dash-pot, $\tau_{ij}=\frac{\eta_j}{G_i},(i=1,2\ ;\ j=2',2,3)$.

Using, Eq. (6) and Eq. (7), we get

$$\left[D^2+\left\{\left(\theta_{12'}+\theta_{13}\right)+\left(\theta_{22}+\theta_{22'}\right)\right\}D+\left\{\theta_{12'}.\theta_{22}+\theta_{13}\left(\theta_{22}+\theta_{22'}\right)\right\}\right]\sigma=G_1\left\{D^2+\left(\theta_{22}+\theta_{22'}\right)D\right\}a \tag{8}$$

Put $R_1=\theta_{12'}+\theta_{13}$, $R_2=\theta_{22}+\theta_{22'}$, $R_3=\theta_{12'}.\theta_{22}$, $R_4=\theta_{13}R_2$ in Eq. (8), we get

$$\left[D^2+\left(R_1+R_2\right)D+\left(R_3+R_4\right)\right]\sigma=G_1\left\{D^2+R_2\,D\right\}a=\left(G_1D^2+G_1\;R_2\,D\right)a \tag{9}$$

The Eq. (9) can be written in terms of differential operator form as:

$$\sum_{n=1}^{2}\alpha_nD^na\left(x,t\right)=\sum_{m=0}^{2}\beta_mD^m\sigma\left(x,t\right) \tag{10}$$

where, the order m and n of sums on right hand side and left hand side in the Eq. (11) depends upon the structure of the mechanical model representing the viscoelastic body. α_n and β_m are the combinations of the material constants and $\alpha_2=G_1,\alpha_1=G_1R_2,\beta_2=1,\ \beta_1=R_1+R_2,\beta_0=R_3+R_4,D\equiv\frac{d}{dt}$.

Eq. (10) is the required differential operator form of constitutive relation for the model for viscoelastic material. The time parameter $'t'$ is introduced into an experimental scheme in dynamic experiments by cyclic deformation of the specimen, frequency ω of the oscillations plays the role of the time factor. The cyclic deformation is the fundamental process of determining the mechanical characteristics. The greatest preference is given to harmonic oscillations. A Harmonic action of the stress/strain produces a corresponding harmonic response in the strain/stress. Let us consider that shearing strain (a), induced in elastic body which can be expressed by a harmonic action as:

$$a = a_0 e^{iwt} \tag{11}$$

where, a_0 is the amplitude, ω is the frequency of oscillations and $'t'$ is the time.

According to Hooke's law, stress $'\sigma'$ is:

$$\sigma = \sigma_0 e^{iwt} \tag{12}$$

where, $\sigma_0 = G_0 \gamma_0$, at t=0

For an elastic body, the strain and stress vary harmonically and there is no lack in the harmonic motion in phase as both have e^{iwt} as a factor. Thus an elastic body responds instantaneously to the external action (strain/stress). The phase shift angle between strain and stress is zero. For an ideal viscous body, the Newton's law of flow to a fluid body is as:

$$\sigma = \eta_0 a_0 i\omega e^{iwt} = \sigma_0 e^{i\left(\omega t + \frac{\pi}{2}\right)} \tag{13}$$

where, η_0 is the viscosity of the body and $\eta_0 a_0 \omega = \sigma_0$ at t=0

Thus, for a viscous deformation, stress advances by the strain by a phase angle $\frac{\pi}{2}$. Thus the phase shift angle for the stress-strain under periodic harmonic deformation for elastic body is zero and for the viscous body, it is $\frac{\pi}{2}$. Therefore the phase shift angle 'δ' for the viscoelastic body must be between zero and $\frac{\pi}{2}$ i.e. $0 < \delta < \frac{\pi}{2}$. The lagging in phase of the strain behind the stress is due to the presence of relaxation processes in the case of viscoelastic body, as phase shift angle δ, is given by $0 < \delta < \frac{\pi}{2}$. Hence,

$$a = a_0 e^{iwt} \tag{14}$$

$$\sigma = \sigma_0 e^{i(wt+\delta)} \tag{15}$$

If we represent the projection of the stress vector on axis of co-ordinates by taking $\sigma' \leftrightarrow x$ and $\sigma'' \leftrightarrow y$, where σ' and σ'' represent that real and imaginary parts respectively. If the strain is initially set harmonically, then the modulus of viscoelastic body with harmonic loading can be written as:

$$\frac{\sigma^*}{a^*} = \frac{\sigma^*}{a'} = \frac{\sigma'}{a'} + i\frac{\sigma''}{a'} = G' + iG'' = G^* \tag{16}$$

The phase angle δ is given as:

$$\tan\delta = \frac{G^*}{G'} \tag{17}$$

In the case of present model (Five-Parameter; two springs $S_1(G_1), S_2(G_2)$; three dash-pots $D_2(\eta_2), D_{2'}(\eta_{2'}), D_3(\eta_3)$ which represents a linear viscoelastic behavior under a given action of loading, the stress is directly proportional to strain. This is also true for time dependent stress and strain relation i.e. for viscoelastic body, the stress is:

$$\sigma(t) = G^* a_0 e^{iwt} \tag{18}$$

where, $a = a_0 e^{iwt}$

Using, the relation $\sigma = Ga$ for an elastic body, the constitutive relation for the physical state representing the five parameter model is given by Eq. (9).

Using Eq. (14), Eq. (16) in Eq. (9), we get

$$\left\{-\omega^2 + \left(R_1 + R_2\right) i\omega + (R_3 + R_4)\right\} G^* a_0 e^{iwt} = G_1 \left(-\omega^2 + R_2 \omega i\right) a_0 e^{iwt} \tag{19}$$

On solving, we get

$$G^*(i\omega) = \frac{G_1 \left(-\omega^2 + R_2 \omega i\right)}{\left(\left(R_3 + R_4 - \omega^2\right) + \left(R_1 + R_2\right) i\omega\right)} \tag{20}$$

$$G' + iG'' = G''(i\omega) = \frac{G_1 \left[\left\{R_2 \left(R_1 + R_2\right) - \left(R_3 + R_4\right) + \omega^2\right\} \omega^2 + \left\{R_2 \left(R_3 + R_4\right) + R_1 \omega^2\right\} \omega i\right]}{A_1} \tag{21}$$

Separate Eq. (21) into real and imaginary parts, we get

$$G' = \frac{G_1 \left[\left\{R_2 \left(R_1 + R_2\right) - \left(R_3 + R_4\right) + \omega^2\right\} \omega^2\right]}{A_1} \tag{22}$$

$$G'' = \frac{G_1 \left[\left\{R_2 \left(R_3 + R_4\right) + R_1 \omega^2\right\} \omega\right]}{A_1} \tag{23}$$

and loss tangent is given by

$$\tan\delta = \frac{G''}{G'} = \frac{R_2 \left(R_3 + R_4\right) + R_1 \omega^2}{\omega \left\{R_2 \left(R_1 + R_2\right) - \left(R_3 + R_4\right) + \omega^2\right\}} = \frac{R_2 \left(R_3 + R_4\right) + R_1 \omega^2}{\omega \left[\omega^2 - \left\{\left(R_3 + R_4\right) - R_2 \left(R_1 + R_2\right)\right\}\right]} \tag{24}$$

To find the values of G'', we put

$$G'' = G_1 \frac{A\omega + B\omega^3}{\left(C - \omega^2\right)^2 + E^2 \omega^2} \tag{25}$$

where

$$A = R_2 \left(R_3 + R_4\right),\ B = R_1,\ C = R_3 + R_4,\ E = R_1 + R_2,\ A_1 = \left(C - \omega^2\right)^2 + E^2 \omega^2$$

The rheological responses are discussed by plotting a graph between G'' and ω and G' verses ω for five parameter model.

To calculate G'' at different frequencies, we assume the following values

Table 1
Material properties

	$G_1 = 1.0$	$G_2 = 1.10$	$\eta_2 = 0.1$	$\eta_{2'} = 0.2$	$\eta_3 = 0.3$
Material	1.0	1.10	0.1	0.2	0.3

Using the values taken in Table 1. , we get

$$R_1 = 8.33, R_2 = 16.5, R_3 = 55, R_4 = 54.945, A = 1814.175, B = 8.33, C = 109.945, E = 24.83$$

Now

$$G^{'} = \frac{G_1\left[\left\{R_2\left(R_1 + R_2\right) - \left(R_3 + R_4\right) + \omega^2\right\}\omega^2\right]}{A_1} \tag{26}$$

where

$$R_1 = 8.33, R_2 = 16.5, R_3 = 55, R_4 = 54.945, A_1 = \left(C - \omega^2\right)^2 + E^2\omega^2$$

Figs. (2-3), has been plotted for five parameter model. It is quite clear from graph, form Fig. 2 and Fig. 3 for $\omega = 7$, there is peak for the graphs $G^{"}$ verse ω and $G^{'}$ verses ω , as the value of ω increases both $G^{"}$ and $G^{'}$ decreases i.e. the exponential decay takes place. However, the decay in Fig. 3 is steeper as compared in Fig.2.

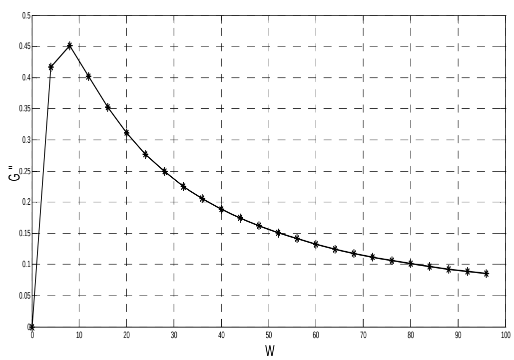

Fig. 2
Variation of $G^{"}$ verses ω for five parameter model.

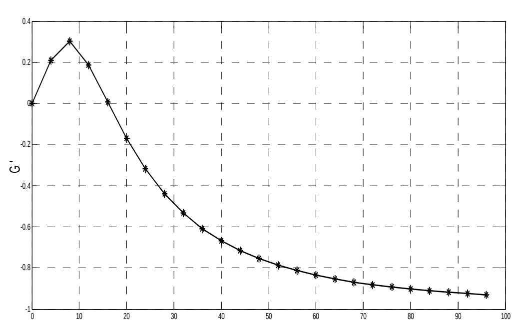

Fig. 3
Variation of $G^{'}$ verses ω for five parameter model.

2.1 Discussion

Moreover, after the removal of stress, the viscous strain in five parameter model is found to remain constant with time. For sufficiently large relaxation time for five parameter models, the stress in the rods will drop to zero. The phase shift angle 'δ' for the viscoelastic body for five parameter model must be between zero and $\frac{\pi}{2}$.

The use of five parameter models is mostly restricted in the field of rock mechanics. Thus, these models can be used in determining the time-dependent behavior of a viscoelastic medium.

3 FORMULATION OF THE PROBLEM

We consider the propagation of waves along the five parameter viscoelastic model and accordingly it as the uniaxial complex modulus $G^*(i\omega)$ that involved the mechanical property. Let the filament be very long and its one end be subjected to a steady state harmonic oscillation condition, then harmonic waves are propagated along the filament with a reduction in the direction of propagation.

The equation of motion is:

$$\frac{\partial^2 \overline{u}}{\partial x^2} = \frac{-\rho\omega^2 \overline{u}}{G^*(i\omega)} \tag{27}$$

where $\overline{u}(x,\omega)$ is the Fourier transformed displacement

Let

$$\left[G^*(i\omega)\right]^{\frac{1}{2}} = G_3(\omega) + iG_4(\omega) \tag{28}$$

Let

$$\frac{1}{c} = \frac{\rho^{\frac{1}{2}} G_3}{G_3^2 + G_4^2} \tag{29}$$

where c is phase velocity.

$$\zeta = \frac{\rho^{\frac{1}{2}} \omega G_4}{G_3^2 + G_4^2} \tag{30}$$

where ζ is the reduction

The solution of Eq. (27) is of the form

$$u(x,t) = Pe^{-\frac{\rho^{\frac{1}{2}}\omega G_4}{\left(G_3^2+G_4^2\right)}x} e^{i\omega\left(t-\frac{\rho^{\frac{1}{2}}\omega G_3}{\left(G_3^2+G_4^2\right)}x\right)} \tag{31}$$

It can also be written as:

$$u(x,t) = Pe^{-\zeta x} e^{i\omega\left(t-\frac{x}{c}\right)} \tag{32}$$

Thus the solution (32) represents the propagation of a harmonic wave moving in the positive x direction with phase velocity c and reduction ζ and the complex constant P is to be determined from boundary conditions.

This method has been widely used to determine high frequency properties, not only with regard of $G^*(i\omega)$ but for stress states as well. Considering the propagation of transient disturbance along the filament and using the Fourier integral of synthesize the velocity from the solution given in Eq. (32).The resulting velocity is given by [24]

$$\dot{u}(x,t)=\int_{-\infty}^{\infty}F(\omega)e^{-\zeta x}e^{i\omega\left(t-\frac{x}{c}\right)}d\omega \tag{33}$$

where $F(\omega)$ is a complex function of frequency to be determined from specified end conditions.

As we have from Eq .(17) and Eq. (22) , $\tan\delta=\frac{G^*}{G'}$ and $G^{"}\left(i\omega\right)=G^{'}(\omega)+iG^{"}(\omega)$

Therefore, we can rewrite the expression for phase velocity c and reduction ζ in Eq. (29) and Eq. (30) as:

$$c=\left(\left|G^*\right|^{\frac{1}{2}}\rho^{\frac{1}{2}}\right)\sec\left(\frac{\delta}{2}\right) \tag{34}$$

$$\zeta=c^{-1}\left\{\omega\tan\left(\frac{\delta}{2}\right)\right\} \tag{35}$$

To give proper dispersion and attenuation of a pulse as it passes from one point of measurement to the other particularly, let

$$\dot{u}(x,t)\big|_{x=x_1}=v_1(t) \tag{36}$$

and

$$\dot{u}(x,t)\big|_{x=x_2}=v_2(t) \tag{37}$$

From Eq. (36), Eq. (37) and the Fourier transform inversion formula, we have

$$F(\omega)=e^{\left[\alpha+(i\omega/c)\right]x_1}\overline{v}_1\left(\omega\right) \tag{38}$$

where,

$$\overline{v}_1\left(\omega\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}v_1(t)e^{-i\omega t}dt \tag{39}$$

Using Eq. (38) in Eq. (33), and evaluating the result at $x=x_2$, using Eq. (37), we get

$$\int_{-\infty}^{\infty}\overline{v}_1(\omega)e^{-\alpha(x_2-x_1)}e^{i\omega\left[t-(x_2-x_1)/c\right]}d\omega=v_2(t) \tag{40}$$

It is to be assumed that the duration of the disturbing pulse is very short that can be represented by a Fourier series rather than the Fourier integral and it would be expected an appropriate procedure for the cases in which very short duration pulses are started by a dispersion effects have broadened the pulse too much. Therefore, a basic time interval is taken which includes the main part of the which is represented by a Fourier series.

Let $\omega = np$, where p is the preselected basic frequency.

Eq.(33) can be written as:

$$\dot{u}(x,t) = \int_{-\infty}^{\infty} F(\omega)e^{-\zeta x}\left\{\cos\omega\left(t-\frac{x}{c}\right)+i\sin\omega\left(t-\frac{x}{c}\right)\right\}d\omega$$

Using the formula for sine and cosine of the difference of two angles, the Fourier series solution corresponding to the Fourier integral Solution given by Eq. (33) is of form

$$\dot{u}(x,t) = \sum_{n=0}^{\infty} e^{-\zeta_n x\left[A_n\cos(npx/c_n)-B_n\sin(npx/c_n)\right]\cos npt} + \sum_{n=0}^{\infty} e^{-\zeta_n x\left[A_n\cos(npx/c_n)-B_n\sin(npx/c_n)\right]\cos npt} \tag{41}$$

where A_n and B_n are unknown constants.

For the sake of convenience, let

$$t-\frac{x}{c} = \left(t-\frac{x}{v}\right)+\left(-\frac{x}{c}+\frac{x}{v}\right). \tag{42}$$

In Eq. (41), replacing t by $t-\dfrac{x}{v}$ and $\dfrac{x}{c}$ by $\dfrac{x}{c}-\dfrac{x}{v}$, we get

$$\dot{u}(x,t) = \sum_{n=0}^{\infty} C_n\cos npt' + \sum_{n=1}^{\infty} D_n\sin npt' \tag{43}$$

where

$$t' = t-\left(\frac{x}{v}\right) \tag{44}$$

$$C_n = e^{-\zeta_n x}\left[A_n\cos\left\{npx(c_n^{-1}-v_n^{-1}\right\}-B_n\sin\left\{npx(c_n^{-1}-v_n^{-1}\right\}\right] \tag{45}$$

and

$$D_n = e^{-\zeta_n x}\left[A_n\sin\left\{npx(c_n^{-1}-v_n^{-1}\right\}+B_n\cos\left\{npx(c_n^{-1}-v_n^{-1}\right\}\right] \tag{46}$$

The velocity v should be taken to be near an average value of propagation of the pulse so that the plus will be maintained within the basic interval of the Fourier series representation. In the Fourier integral representation, the velocity is measured at two locations, x_1 and x_2 , as given by Eq. (36) and Eq. (37). To evaluate Eq. (43) at $x = x_1$, using Eq. (36),and in the usual way the coefficients are given by

$$C_{n1} = \frac{p}{\pi}\int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\cos npt'dt', \qquad n=1,2,\ldots\ldots \tag{47}$$

$$D_{n1} = \frac{p}{\pi}\int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\sin npt'dt', \qquad n=1,2,\ldots\ldots \tag{48}$$

The coefficients A_n and B_n are obtained from Eq.(45) and Eq. (46) as:

$$A_n = e^{\zeta_n x_1}\left[C_{n1}\cos\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}+D_{n1}\sin\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\right] \tag{49}$$

and

$$B_n = e^{\zeta_n x_1}\left[-C_{n1}\sin\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}+D_{n1}\cos\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\right] \tag{50}$$

Thus Eqs. (43)-(46) specify the response with A_n and B_n given by Eqs. (47)-(50).

At $x = x_2$, using Eq.(37) to synthesize the plus into Fourier series

$$\dot{u}(x,t)\Big|_{x=x_2} = \sum_{n=0}^{\infty} P_n \cos npt' + \sum_{n=1}^{\infty} Q_n \sin npt' \tag{51}$$

where

$$P_n = \frac{p}{\pi}\int_{-\pi/p}^{\pi/p} v_2\left(t'\right)\cos npt'dt', \qquad n = 1,2,\ldots\ldots \tag{52}$$

$$Q_n = \frac{p}{\pi}\int_{-\pi/p}^{\pi/p} v_2\left(t'\right)\sin npt'dt', \qquad n = 1,2,\ldots\ldots \tag{53}$$

To evaluate the solution of Eq. (43) at $x = x_2$ and equate the result to Eq. (51).Using Eqs. (45)-(50), (52) and (53) ,the term by term equivalence of these two forms gives the two general relations

$$\begin{aligned}
&\begin{bmatrix}\cos\left\{npx_2(c_n^{-1}-v_n^{-1}\right\}\sin\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\\ -\sin\left\{npx_2(c_n^{-1}-v_n^{-1}\right\}\cos\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\end{bmatrix}\int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\sin npt'dt'\\
&+\begin{bmatrix}\cos\left\{npx_2(c_n^{-1}-v_n^{-1}\right\}\cos\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\\ -\sin\left\{npx_2(c_n^{-1}-v_n^{-1}\right\}\sin\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\end{bmatrix}\int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\sin npt'dt', \qquad n = 1,2,\ldots\ldots\\
&= e^{-\zeta_n(x_1-x_2)}\int_{-\pi/p}^{\pi/p} v_2\left(t'\right)\cos npt'dt',
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
&\begin{bmatrix}\sin\left\{npx_2(c_n^{-1}-v_n^{-1}\right\}\sin\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\\ +\cos\left\{npx_2(c_n^{-1}-v_n^{-1}\right\}\cos\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\end{bmatrix}\int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\sin npt'dt'\\
&+\begin{bmatrix}\sin\left\{npx_2(c_n^{-1}-v_n^{-1}\right\}\cos\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\\ -\cos\left\{npx_2(c_n^{-1}-v_n^{-1}\right\}\sin\left\{npx_1(c_n^{-1}-v_n^{-1}\right\}\end{bmatrix}\int_{-\pi/p}^{\pi/p} v_1\left(t'\right)\cos npt'dt', \qquad n = 1,2,\ldots\ldots\\
&= e^{-\zeta_n(x_1-x_2)}\int_{-\pi/p}^{\pi/p} v_2\left(t'\right)\sin npt'dt',
\end{aligned} \tag{55}$$

Considering with $v_1\left(t^{'}\right)$ and $v_2\left(t^{'}\right)$,the particle velocities at two different locations on the filament are known from experimental measurements, Eqs.(54) and (55) include two equations in two unknowns, c_n and ζ_n , for each value of n .Then using Eqs. (29) and (30),we get

$$G_{3n} = \rho^{\frac{1}{2}} c_n \left[\left(\zeta_n^2 c_n^2 / \omega_n^2\right) + 1\right]^{-1} \tag{56}$$

and

$$G_{4n} = \rho^{\frac{1}{2}} \zeta_n c_n^2 \omega_n^{-1} \left[\left(\zeta_n^2 c_n^2 / \omega_n^2\right) + 1\right]^{-1} \tag{57}$$

where $\omega_n = np$, G_{3n} and G_{4n} are values of G_3 and G_4 right to the n^{th} frequency component. The real and imaginary parts of the complex modulus, $G^*(i\omega)$, may be then found directly from Eq.(28) as:

$$G_n^{'} = \left(G_{3n}\right)^2 - \left(G_{4n}\right)^2 \tag{58}$$

and

$$G_n^{''} = 2G_{3n}G_{4n} \tag{59}$$

The described procedure for determining $G^*(i\omega)$ in the high frequency range has not been explicitly used. The simplest assumptions are to be made taking particular forms for phase velocity and reduction. The simplest assumption is

$$c = \text{Constant} \tag{60}$$

and

$$\tan\delta = \text{Constant} \tag{61}$$

where the reduction can be found from Eq. (35).For this, the frequency range of relevance is narrow enough, then $G^*(i\omega)$ is effectively constant in this region and therefore c and $\tan\delta$ can then be found from a slight modification of the described procedure.

3.1 Wave propagation for the non-homogeneous cases

The stress strain relation for five parameter viscoelastic model is given by the Eq. (6).

The equation of motion and strain-displacement relations are given by

$$\frac{\partial\sigma}{\partial x} = \rho\frac{\partial^2 U}{\partial t^2} \tag{62}$$

$$a = \frac{\partial U}{\partial x} \tag{63}$$

where, $\rho = \rho(x)$ is the variable density of the material

Differentiating Eq. (62) w.r.t x , we get

$$-\frac{1}{\rho^2}\frac{\partial\rho}{\partial x}\frac{\partial\sigma}{\partial x}+\frac{1}{\rho}\frac{\partial^2\sigma}{\partial x^2}=u_{,xtt} \tag{64}$$

Differentiating Eq. (63) w.r.t. t ,we get

$$\frac{\partial a}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) \tag{65}$$

Again differentiating w.r.t. t ,

$$\frac{\partial^2 a}{\partial t^2}=\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\right\}=u_{,ttx}=u_{,xtt} \tag{66}$$

Using Eq. (64) and Eq. (66), Eq. (8) gives

$$\sigma_{,ttt}+\left\{\frac{G_1}{\eta_{2'}}+\frac{G_1}{\eta_3}+\frac{G_2}{\eta_{2'}}+\frac{G_2}{\eta_2}\right\}\sigma_{,tt}+\left\{\frac{G_1}{\eta_{2'}}\frac{G_2}{\eta_2}+\frac{G_1}{\eta_3}\left(\frac{G_2}{\eta_{2'}}+\frac{G_2}{\eta_2}\right)\right\}\sigma_{,t}=\frac{G_1}{\rho}\left\{\frac{\partial}{\partial t}+\frac{G_2}{\eta_{2'}}\left(1+\frac{\eta_{2'}}{\eta_2}\right)\right\}\left\{\frac{\partial}{\partial x}-\frac{\rho'}{\rho}\right\}\frac{\partial\sigma}{\partial x} \tag{67}$$

4 SOLUTION OF THE PROBLEM

Let the solution $\sigma(x,t)$ of Eq. (67) be represented by the series [19]

$$\sigma(x,t)=\sum_{n=0}^{\infty}A_n(x)F_n\{t-h(x)\}, \qquad A_0\neq 0 \tag{68a}$$

where

$$F'_n=F_{n-1} \quad \text{where, n = 1, 2, 3............} \quad \text{with} \quad F_{n,t}=F_{n-1} \text{ and } F_{n,x}=-h_{,x}F_{n-1} \tag{68b}$$

and for $n<0$ assume that $A_n=0$ and the derivatives of σ may be obtained by term-wise differentiation of Eq. (68a), the prime in Eq. (69) denotes differentiation with respect to the argument concerned, and by using Eq. (68b) and Eq.(11) we relate all $F_n^{'}$s to F_0 by successive integrations.

The Solution of Eq. (67) in the form of Eq. (68) can be obtained by taking a phase function $h(x)$, $h(x)$ satisfies the Eikonal equation of geometrical optics [20]

$$\left(\frac{dh(x)}{dx}\right)^2=\frac{\rho}{G_1}=\frac{1}{c^2} \tag{69}$$

where c=c(x) is the variable wave speed for viscoelastic harmonic waves in a medium whose modulus of elasticity G_1 .Using, Eq.(10), Eq.(11) and the successive derivatives of $\sigma(x,t)$ *w.r.t.* *'t'* and *'x'* in Eq.(67), we get the amplitude function satisfy the equation

$$2h^{'}(x)A^{'}_n(x)+\left\{\rho\left(\frac{1}{\eta_{2'}}+\frac{1}{\eta_3}\right)-(log\rho)_{,x}h^{'}(x)+h''(x)\right\}A_n(x)=Q_n, \qquad (n=0,1,2\ldots) \tag{70}$$

where

$$Q_n = A''_{n-1} - \left\{ (log\rho)_{,x} + 2\frac{G_2}{\eta_{2'}}\left(1+\frac{\eta_2'}{\eta_2}\right)h' \right\} A'_{n-1} +$$
$$\left\{ \rho\left(\frac{G_2}{\eta_2\eta_2'} + \frac{G_2}{\eta_2'\eta_3}\left(1+\frac{\eta_2'}{\eta_2}\right)\right) + \frac{G_2}{\eta_2'}\left(1+\frac{\eta_2'}{\eta_2}\right)h'' - \frac{G_2}{\eta_2'}\left(1+\frac{\eta_2'}{\eta_2}\right)(log\rho)_{,x} h' \right\} A_{n-1} +$$
$$\frac{G_2}{\eta_2'}\left(1+\frac{\eta_2'}{\eta_2}\right)A''_{n-2} - \frac{G_2}{\eta_2'}\left(1+\frac{\eta_2'}{\eta_2}\right)(log\rho)_{,x} A'_{n-2} + \frac{G_2}{\eta_2}A''_{n-2} - \frac{G_2}{\eta_2}(log\rho)_{,x} A'_{n-2}$$

Since the wave is travelling along x-axis, therefore, integrating Eq. (70), we get

$$h(x) = h(0) \pm \int_0^x \frac{ds}{c(s)} \tag{71}$$

Since Eq. (70) is a first order linear differential equation in $A_n(x)$.Therefore, the general solution of Eq. (70) can be obtained as:

$$A_n(x) = A_n(0)\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}} exp\left\{\mp\int_0^x m(s)ds\right\} \pm \frac{1}{2}\int_0^x c(s)\left\{\frac{l(x)}{l(s)}\right\}^{\frac{1}{2}} exp\left\{\pm\int_x^z m(z)dz\right\} Q_n^{\pm}(s)ds, \qquad (n = 0,1,2..\tag{72}$$

where, $l(x) = \rho c$ and $m(x) = \frac{\rho c}{2}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$.

The plus '$+$' sign is associated with wave traveling in the positive direction of *x* and the minus '$-$' sign is associated with the waves travelling in the negative direction of *x*.

Let an impulse of magnitude σ_0 suddenly applied at the end $x = 0$ of the rod and thereafter steadily maintained, that is

$$\sigma(0,t) = \sigma_0 H(t) \tag{73}$$

From Eq. (68) and Eq. (73), we have

$$\sum_{n=0}^{\infty} A_n(0) F_n\{t - h(0)\} = \sigma_0 H(t) \tag{74}$$

Thus we choose [21]

$$A_n(0) = \begin{vmatrix} \sigma_0 \cdots\cdots\cdots if\ n = 0 \\ 0 \cdots\cdots\cdots if\ n < 0\, or\, n > 0 \end{vmatrix} \tag{75}$$

$$h(0) = 0 \ \ and \ \ F_0 = H(t) \tag{76}$$

The solution of Eq. (67), for the waves travelling in the positive direction of x is generated by boundary stress Eq. (73), is

$$\sigma(x,t)=\sum_{n=0}^{\infty}A_n(x)\frac{\{t-h(x)\}^n}{n!}H\{t-h(x)\}=\sum_{n=0}^{\infty}A_n(x)\frac{\{t-h(x)\}^n}{n!}H\left\{t-\int_0^x\frac{ds}{c(s)}\right\}$$

where

$$h(x)=\int_0^x\frac{ds}{c(s)}$$

where, $A_n(x)$ are given recursively by Eq. (72) (with upper signs) in combination with Eq. (75).

The first–term approximation leads to Eq. (72) is

$$\sigma(x,t)=\sigma_0\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}}exp\left\{-\int_0^x m(s)\,ds\right\}H\left\{t-\int_0^x\frac{ds}{c(s)}\right\}$$

The Eq. (76) represents a transient stress wave which starts from the end ‘x=0’ with amplitude’ $'\sigma_0'$ and moves in the positive direction of ‘*x*’ with velocity c(x). Hence, it is modulated by the factor

$$\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}}exp\left\{-\int_0^x m(s)\,ds\right\} \tag{77}$$

Further terms in the approximate solution may be obtained recursively from Eq. (72). The solution of Eq. (74) applies until the wave moving in the positive direction of ‘*x*’ strikes either an interface (in the case of a composite rod) or at end (in the case of a finite rod).

5 VALIDITY OF FIVE PARAMETER MODEL

For the sake of concreteness and for studying the qualitative effect of non-homogeneity on the harmonic wave propagation in non-homogeneous five parameter viscoelastic rods, it is assumed that density $'\rho'$, rigidity $'G'$ and viscosity $'\eta'$ of the specimen i.e. rod are space dependent and obey the laws

$$\rho=\rho_0(1+\cos\alpha_1 x)\ \ ,G=G_0(1+\cos\alpha_2 x),\eta=\eta_0(1+\cos\alpha_3 x) \tag{78}$$

If, $\alpha_1\ge\alpha_2\ge\alpha_3$ i.e. density $\ge$ rigidity $\ge$ viscosity

Case 1.

When, $\alpha_1=\alpha_2=\alpha_3$, then from Eq. (25), we get

$$\rho=\rho_0(1+\cos\alpha x)\ \ ,G=G_0(1+\cos\alpha x),\eta=\eta_0(1+\cos\alpha x) \tag{79}$$

Therefore, from Eikonal equation of geometric optics

$$\left(\frac{dh(x)}{dx}\right)^2=\frac{\rho}{G_1}=\frac{\rho_0(1+\cos\alpha x)}{G_{01}(1+\cos\alpha x)}=\frac{\rho_0}{G_{01}}=\frac{1}{c_0{}^2}=\text{constant}. \tag{80}$$

or

$$c_0 = \sqrt{\frac{G_{01}}{\rho_0}} \tag{81}$$

Since, the exponential variation of modulus of rigidity G and density ρ is similar, therefore sound speed is constant i.e. non-homogeneous has no effect on speed and phase of the wave is given $h(x) = \frac{x}{c_0}$. So it becomes the case of semi non-homogeneous medium (a medium when characteristics are space dependent while the speed is independent of space variable).

The amplitude function $A_n(x)$ satisfies the equation

$$2h'(x)A'_n(x) + \left\{\rho_0\left(\frac{1}{\eta_{02'}}+\frac{1}{\eta_{03}}\right) - 2\alpha h'(x) + h''(x)\right\}A_n(x) = Q'_n, \qquad (n = 0,1,2\ldots\ldots\ldots) \tag{82}$$

$$\begin{aligned} Q_n &= A''_{n-1} - \left\{\alpha\frac{\sin\alpha x}{(1+\cos\alpha x)} + 2\frac{G_{02}}{\eta_{02'}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)h'\right\}A'_{n-1} - \\ &\left\{\rho_0\left(\frac{G_{02}}{\eta_{02}\eta_{02'}} + \frac{G_2}{\eta_{02'}\eta_{03}}\left(1+\frac{\eta_{02'}}{\eta_2}\right)\right) + \frac{G_{02}}{\eta_{02'}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)h'' - \frac{G_{02}}{\eta_{02'}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)\alpha\frac{\sin\alpha x}{(1+\cos\alpha x)}h'\right\}A_{n-1} + \\ &\frac{G_{02}}{\eta'_{02}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)A''_{n-2} - \frac{G_{02}}{\eta_{02'}}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)\alpha\frac{\sin\alpha x}{(1+\cos\alpha x)}A'_{n-2} \end{aligned} \tag{83}$$

As the amplitude function is given by Eq.(72), For this case

$$l(x) = \sqrt{\rho_0 G_{10}}\,(1+\cos\alpha x), m(x) = \frac{\sqrt{\rho_0 G_{01}}}{2}\left(\frac{1}{\eta_{02'}}+\frac{1}{\eta_{03}}\right) = m_0, \int_0^x m(x)dx = m_0 x \tag{84}$$

Hence

$$A_n(x) = A_n(0)\sqrt{\frac{(1+\cos\alpha x)}{2}}\exp\left\{\mp\int_0^x m_0 ds\right\} \pm \frac{1}{2}\int_0^x c_0\left(\frac{1+\cos\alpha x}{1+\cos\alpha s}\right)\exp\left\{\pm\int_x^s m_0 dz\right\}Q'^{\pm}_n(s)\,ds \tag{85}$$

For the case of the value of first term approximation, the stress function is given by

$$\sigma(x,t) = \sigma_0\sqrt{\frac{(1+\cos\alpha x)}{2}}exp\left\{-\int_0^x m_0 ds\right\}H\{t-h(x)\} \tag{86}$$

The harmonic stress wave which starts from the end $x = 0$ with amplitude σ_0 and moves with constant velocity $c_0 = \sqrt{\frac{G_{01}}{\rho_0}}$ in the positive direction of x is modulated by the factor

$$\sqrt{\frac{(1+\cos\alpha x)}{2}}exp\left\{-\int_0^x m_0 ds\right\} \tag{87}$$

Further terms in the approximation solution may be obtained from Eq. (85).

Case 2.
$\alpha_1 > \alpha_2 > \alpha_3$ i.e. density $>$ rigidity $>$ viscosity, then from Eq. (78), we get

$$\rho = \rho_0(1+\cos\alpha_1 x) \ , G = G_0(1+\cos\alpha_2 x), \eta = \eta_0(1+\cos\alpha_3 x)$$

From Eikonal equation of geometric optics

$$\left(\frac{dh(x)}{dx}\right)^2 = \frac{\rho}{G_1} = \frac{\rho_0\left(1+\cos\alpha_1 x\right)}{G_{10}\left(1+\cos\alpha_2 x\right)} = \frac{1}{c^2} \tag{88}$$

Hence

$$c = \sqrt{\frac{G_{01}\left(1+\cos\alpha_2 x\right)}{\rho_0\left(1+\cos\alpha_1 x\right)}} \tag{89}$$

The amplitude function $A_n\left(x\right)$ satisfies the equation

$$2h^{'}\left(x\right)A^{'}_{n}\left(x\right)+\left\{\rho_0\frac{\left(1+\cos\alpha_1 x\right)}{\left(1+\cos\alpha_3 x\right)}\left(\frac{1}{\eta_{02'}}+\frac{1}{\eta_{03}}\right)-\alpha_1\frac{\sin\alpha_1 x}{\left(1+\cos\alpha_1 x\right)}h^{'}(x)+h''(x)\right\}A_n\left(x\right)=Q^{"}_{n}, \tag{90}$$

$(n=0,1,2\ldots)$

where

$$Q^{"}_{n} = A''_{n-1} - \left\{\alpha_1\frac{\sin\alpha_1 x}{\left(1+\cos\alpha_1 x\right)} + 2\frac{G_{02}\left(1+\cos\alpha_2 x\right)}{\eta_{02'}\left(1+\cos\alpha_3 x\right)}\left(1+\frac{\eta_{02'}}{\eta_{02}}\right)h'\right\}A'_{n-1} - \left\{\begin{array}{l}\rho_0\left(1+\cos\alpha_1 x\right)\left(\dfrac{G_{02}\left(1+\cos\alpha_2 x\right)}{\eta_{02}\eta_{02'}\left(1+\cos\alpha_3 x\right)^2}+\dfrac{G_{02}\left(1+\cos\alpha_2 x\right)}{\eta_{02'}\eta_{03}\left(1+\cos\alpha_3 x\right)^2}\left(1+\dfrac{\eta_{02'}}{\eta_{02}}\right)\right)+\\ \dfrac{G_{02}\left(1+\cos\alpha_2 x\right)}{\eta_{02'}\left(1+\cos\alpha_3 x\right)}\left(1+\dfrac{\eta_{02'}}{\eta_{02}}\right)h^{"}-\dfrac{G_{02}\left(1+\cos\alpha_2 x\right)}{\eta_{02'}\left(1+\cos\alpha_3 x\right)}\left(1+\dfrac{\eta_{02'}}{\eta_{02}}\right)\alpha\dfrac{\sin\alpha x}{\left(1+\cos\alpha x\right)}h'\end{array}\right\}A_{n-1} + \frac{G_{02}\left(1+\cos\alpha_2 x\right)}{\eta_{02'}\left(1+\cos\alpha_3 x\right)}\left(1+\frac{\eta_2^{'}}{\eta_2}\right)A^{"}_{n-2} - \frac{G_{02}\left(1+\cos\alpha_2 x\right)}{\eta_{02'}\left(1+\cos\alpha_3 x\right)}\alpha\frac{\sin\alpha x}{\left(1+\cos\alpha x\right)}A^{'}_{n-2} \tag{91}$$

$(n=0,1,2\ldots\ldots\ldots)$

and Amplitude function $A_n(x)$ is given by Eq. (72).

For this case

$$l(x) = \sqrt{\rho_0 G_0\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)} = l_1(x)$$

and

$$m(x)=\frac{\sqrt{\rho_0 G_0\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)}}{2\left(1+\cos\alpha_3 x\right)}\left\{\frac{1}{\eta_{02'}}+\frac{1}{\eta_{03}}\right\}=m_1(x)$$

$$\int_0^x m(x)dx=\frac{\sqrt{G_{01}\rho_0}}{2}\left\{\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right\}\int_0^x\frac{\sqrt{\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)}}{\left(1+\cos\alpha_3 x\right)}dx$$

Therefore

$$A_n(x)=A_n(0)\left[\frac{\sqrt{\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)}}{2}\right]^{\frac{1}{2}} exp\left\{\mp\int_0^x m_1(s)\,ds\right\}\pm\frac{1}{2}\int_0^x c(s)\left\{\frac{l_1(x)}{l_1(s)}\right\}^{\frac{1}{2}} exp\left\{\pm\int_x^z m_1(z)\,dz\right\} Q_n^{''\pm}(s)\,ds \tag{92}$$

For the case of the value of first term approximation, the stress function is given by

$$\sigma(x,t)=\sigma_0\left[\frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{2}\right]^{\frac{1}{2}} exp\left\{-\frac{\sqrt{G_{01}\rho_0}}{2}\left\{\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right\}\int_0^x\frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{(1+\cos\alpha_3 x)}dx\right\}H\left\{t-h(x)\right\} \tag{93}$$

where

$$h(x)=\frac{1}{\left(\alpha_1-\alpha_2\right)}\sqrt{\frac{\rho_0}{G_{10}}}e^{(\alpha_1-\alpha_2)x}$$

Eq. (93) gives a harmonic stress wave which starts from the end $x=0$ with amplitude σ_0 and moves with velocity, $c=\sqrt{\frac{G_{01}\left(1+\cos\alpha_2 x\right)}{\rho_0\left(1+\cos\alpha_1 x\right)}}$ in the positive direction of x and is modulated by the factor

$$\left[\frac{\sqrt{\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)}}{2}\right]^{\frac{1}{2}} exp\left\{-\frac{\sqrt{G_{01}\rho_0}}{2}\left\{\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right\}\int_0^x\frac{\sqrt{\left(1+\cos\alpha_1 x\right)\left(1+\cos\alpha_2 x\right)}}{\left(1+\cos\alpha_3 x\right)}dx\right\} \tag{94}$$

6 NUMERICAL ANALYSIS

To see qualitative effect of non-homogeneity on the harmonic wave propagation in non-homogeneous five parameter viscoelastic rod, a graph is plotted between σ/σ_0 and x for Eq. (86) (semi-homogeneous case), by taking $\alpha=\left(4,\ 3,\ 0,-3,-4\right)$. The material properties are as shown in Table.2

Table 2
Material properties

	ρ_0	G_{01}	G_{02}	$G_{02}^{'}$	η_{03}	η'_{02}
Material	2.0	1.8	1.6	1.4	1.5	1.3

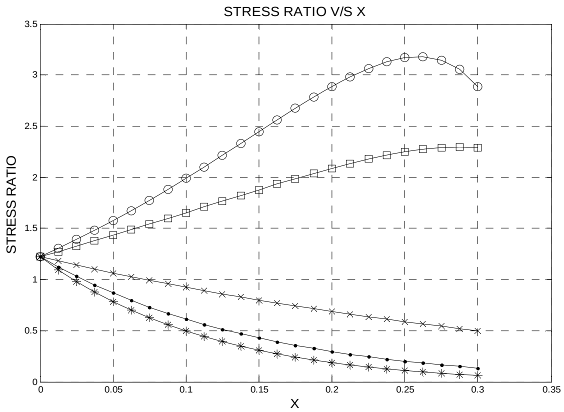

Fig. 4
(Taking α = 4, 3, 0,-3,-4).

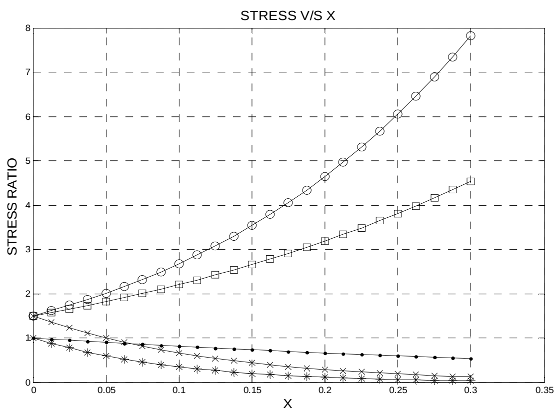

Fig. 5
(Taking α = 4, 3, 0,-3,-4).

For non- homogeneous case, Figure is plotted between σ/σ_0 v/s x by taking Eq.(93) as by taking $\alpha_1 = 2.2, \alpha_2 = 2.4, \alpha_3 = 2.6$ in the Eq. (93)

$$\frac{\sigma(x,t)}{\sigma_0} = \left[\frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{2}\right]^{\frac{1}{2}} exp\left\{-\frac{\sqrt{G_{01}\rho_0}}{2}\left\{\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right\}\int_0^x \frac{\sqrt{(1+\cos\alpha_1 x)(1+\cos\alpha_2 x)}}{(1+\cos\alpha_3 x)}dx\right\} H\{t-h(x)\}$$

7 CONCLUSIONS

When the density, rigidity and viscosity all are equal for the first material specimen, the sound speed is constant i.e. non-homogeneous has no effect on speed and phase of the wave is given $h(x) = \frac{x}{c_0}$. So it becomes the case of semi non-homogeneous medium (a medium when characteristics are space dependent while the speed is independent of space variable). The harmonic speed will be equal to $c = \sqrt{\frac{G_0}{\rho_0}}$.

When the density, rigidity and viscosity are not equal for the second material specimen, the speed of sound is given as $c = \sqrt{\frac{G_{01}(1+\cos\alpha_2 x)}{\rho_0(1+\cos\alpha_1 x)}}$.

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