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Vibration Analysis of Multi-Step Bernoulli-Euler and Timoshenko Beams Carrying Concentrated Masses

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ABSTRACT

In this paper, vibration analysis of multiple-stepped Bernoulli-Euler and Timoshenko beams carrying point masses is presented analytically for various boundary conditions. Each attached element is considered to have both translational and rotational inertias. The method of solution is "transfer matrix method" which is based on the changes in the vibration modes at the vicinity of any discontinuity in geometrical and natural parameters; these changes are shown by transfer matrices depended on the geometry of each step or value of the translational and rotational inertias of each attached element. First, natural frequencies and corresponding normal mode shapes are obtained by implementation of the compatibility conditions and external boundary conditions; Then, the precision of the proposed method is checked by comparison of the results with other exact solutions; Finally, the effect of the translational and rotational inertias and position of the attached elements on the natural frequencies of multi-stepped beams are investigated for various boundary conditions.

Keywords: Bernoulli-Euler beam; Timoshenko beam; Multi-Step beam; Concentrated mass; Rotary inertia

1 INTRODUCTION

STUDYING the dynamic characteristics of systems with flexible links or components is an essential research endeavour that can provide successful design of mechanisms, robots, machines, and structures. Extensive researches have been carried out with regard to the vibration analysis of the structures carrying concentrated masses at arbitrary positions. Chen [1] introduced the mass by the Dirac delta function and solved analytically the problem of a vibrating simply supported beam carrying a concentrated mass at its middle section. Laura et al. [2] studied the cantilever beam carrying a lumped mass at the top, introducing the mass in the boundary conditions. Laura et al. [3] used Rayleigh–Ritz method to analyze beams subjected to axial forces and carrying concentrated masses. Gurgoze [4, 5] used the normal mode summation technique to determine the fundamental frequency of a cantilever beam carrying masses and rotational springs. Liu et al. [6] used the Laplace transformation technique to formulate the frequency equation for the beams with elastically restrained ends, carrying concentrated masses. Rosa et al. [7] investigated dynamic behaviour of the beams with elastic ends carrying a concentrated mass. Rossit and Laura [8] presented a solution for vibration analysis of a cantilever beam with a spring-mass system attached on the free end. In all studies mentioned above, authors used Bernoulli-Euler beam theory to model simple structures, which is reliable just for slender beams. In order to increase accuracy and reliability of the studies, especially for the short beams, Rao et al. [9] used coupled displacement field method to study about natural frequencies of a Timoshenko

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beam with a central point mass. Rossit and Laura [10] extended their previous research [8] for Timoshenko beam theory.

In most of the studies mentioned above, the influence of the rotary inertia of the attached mass is not taken into account. Laura et al. [11] considered the rotary inertia of concentrated masses attached to the slender beams and plates and obtained fundamental frequencies of the coupled systems by means of the Rayleigh–Ritz and Dunkerley methods. Rossi and Laura [12] focused on vibrations of a Timoshenko beam clamped at one end and carrying a finite mass at the other. They considered both the translational and rotational inertia of the attached mass. Chang [13] studied a simply supported beam carrying a mass on its middle point and considered its rotary inertia. He determined the natural frequencies and normal modes of the system, but he kept the position of the mass fixed. Maiz et al. [14] presented an exact solution for the transverse vibration of Bernoulli–Euler beam carrying point masses and taking into account their rotary inertia. Lin [15] used numerical assembly method to determine the exact natural frequencies and mode shapes of the multi-span Timoshenko beam carrying a number of various concentrated elements including point masses, rotary inertias, linear springs, rotational springs and spring–mass systems. Demirdag and Murat [16] and Demirdag and Yesilce [17] used fuzzy neural network and differential transform method respectively to study about elastically supported Timoshenko columns with tip mass having rotary inertia. Guitirrez et al. [18] studied stepped Timoshenko beam, elastically restrained at one end and carrying a mass having rotary inertia at the other one.

Stepped beam are widely used in various engineering fields, such as turning shaft, robot arm and tall building, etc. The free vibration analysis of one or multiple stepped beams are investigated by many researchers and plentiful achievements are obtained. No attempt will be made here to present a bibliographical account of previous work in this area. Just a few selective recent papers [19-22] which provide further references on the subject are quoted.

The purpose of this study is to present a general solution for the vibration analysis of multiple-stepped Bernoulli-Euler and Timoshenko beams, carrying concentrated masses having rotary inertia at arbitrary points, for various boundary conditions. Effect of the value of translational and rotational inertias of attached masses and their positions on the frequencies of vibration will be studied in this paper for both uniform and multiple-stepped beams.

2 BERNOULLI-EULER BEAM THEORY

Bernoulli-Euler beam theory is the simplest model which can be used to study about static and dynamic behaviour of structures. The accuracy and reliability of this model are acceptable just for slender beams. Because of simplicity, this model is investigated before the Timoshenko one.

2.1 Governing equation

As shown in Fig. 1, an elastic beam, having some discontinuities located on its domain like steps or concentrated masses is considered. Translational and rotational inertias of the attached masses are indicated by notations "m" and "j", respectively. Also each step is modeled by two parameters (ξ & μ) which would be defined later.

Governing equation of free bending vibration analysis of a bare beam can be written as [23]

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]+\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}=0, \tag{1}$$

where w(x,t), A(x), I(x), ρ and E are vertical displacement, cross-sectional area, area moment of inertia about the neutral axis, mass density and Young's modulus of material, respectively. Let indicate any property of the beam at *i*th segment with index "*i*"; Notice that as shown in Fig. 1, there is no difference between steps and concentrated masses in numbering sub-beams. For *i*th segment of the beam, Eq. (1) can be rewritten as:

$$EI_{(i)}\frac{\partial^4 w_i(x,t)}{\partial x^4}+\rho A_{(i)}\frac{\partial^2 w_i(x,t)}{\partial t^2}=0 \quad i=1,2,...,n+1, \tag{2}$$

where n is the number of all discontinuities (including both steps and concentrated masses). The vertical displacement $w_i(x,t)$ can be assumed as the product of the function $v_i(x)$ depended on the spatial coordinate x and a time dependent harmonic function as:

$$w_i(x,t) = Lv_i(x)e^{j\omega t}, \tag{3}$$

In which ω is the natural frequency of vibration and j is the imaginary unit ($j^2 = -1$). Substitution of the separated form provided by Eq. (3) into the governing Eq. (2) yields the following differential equation:

$$v_i'''' - \lambda_i^4 v_i = 0, \tag{4}$$

where the prime indicates the derivative with respect to the dimensionless spatial variable (ζ) and following dimensionless parameters are defined:

$$\zeta = \frac{x}{L} \quad \lambda_i^4 = \frac{\rho A_i L^4 \omega^2}{EI_i} = \frac{A_i I_1}{A_1 I_i}\lambda_1^4. \tag{5}$$

Solution of Eq. (4) can be found as:

$$v_i(\zeta) = A_i \cosh(\lambda_i \zeta) + B_i \sinh(\lambda_i \zeta) + C_i \cos(\lambda_i \zeta) + D_i \sin(\lambda_i \zeta), \tag{6}$$

in which A_i-D_i are unknown constant coefficients.

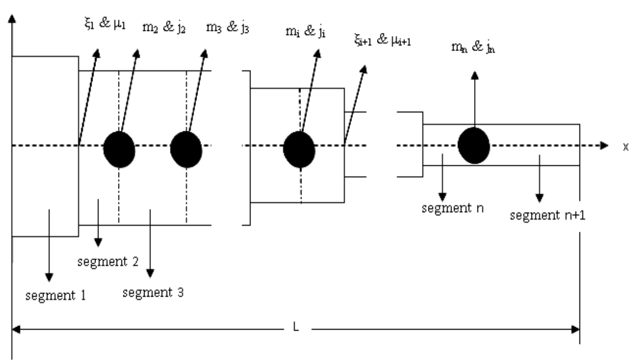


Fig. 1
Geometry and parameters of the system.

2.2 Compatibility conditions

In the vicinity of ith discontinuity, Eq. (6) can be written as:

$$\begin{aligned} v_i(\zeta - e_{i-1}) &= A_i \cosh\left[\lambda_i(\zeta - e_{i-1})\right] + B_i \sinh\left[\lambda_i(\zeta - e_{i-1})\right] + C_i \cos\left[\lambda_i(\zeta - e_{i-1})\right] + D_i \sin\left[\lambda_i(\zeta - e_{i-1})\right], \\ v_{i+1}(\zeta - e_i) &= A_{i+1} \cosh\left[\lambda_{i+1}(\zeta - e_i)\right] + B_{i+1} \sinh\left[\lambda_{i+1}(\zeta - e_i)\right] + C_{i+1} \cos\left[\lambda_{i+1}(\zeta - e_i)\right] + D_{i+1} \sin\left[\lambda_{i+1}(\zeta - e_i)\right] \end{aligned} \tag{7}$$

where e_i is dimensionless spatial coordinate of ith discontinuity.

2.2.1 Stepped section

Suppose that ith discontinuity is a step, the compatibility conditions at the stepped section can be assumed as continuity of vertical displacement, slope, bending moment and shear force. According to the definitions of the bending moment and shear force in Bernoulli-Euler beam theory as [23]

$$M_i = \frac{EI_i}{L} v_i'' \quad V_i = \frac{EI_i}{L^2} v_i''' , \tag{8}$$

Compatibility conditions at the stepped section can be rewritten as:

$$v_i = v_{i+1} \quad v_i' = v_{i+1}' \quad v_i'' = \xi_i\, v_{i+1}'' \quad v_i''' = \xi_i\, v_{i+1}''' , \tag{9}$$

where

$$\xi_i = \frac{I_{i+1}}{I_i}. \tag{10}$$

Using Eqs. (7) and (9), the constant coefficients after ith discontinuity are related to those before it as:

$$\{A_{i+1} \quad B_{i+1} \quad C_{i+1} \quad D_{i+1}\}^T = T^{(i)}\{A_i \quad B_i \quad C_i \quad D_i\}^T , \tag{11}$$

where

$$T^{(i)} = \left[P^{(i)}\right]^{-1} Q^{(i)} \qquad P^{(i)} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_{i+1} & 0 & \lambda_{i+1} \\ \lambda_{i+1}{}^2\xi_i & 0 & -\lambda_{i+1}{}^2\xi_i & 0 \\ 0 & \lambda_{i+1}{}^3\xi_i & 0 & -\lambda_{i+1}{}^3\xi_i \end{bmatrix}$$

$$Q^{(i)} = \begin{bmatrix} \cosh(\lambda_i\delta_i) & \sinh(\lambda_i\delta_i) & \cos(\lambda_i\delta_i) & \sin(\lambda_i\delta_i) \\ \lambda_i \sinh(\lambda_i\delta_i) & \lambda_i \cosh(\lambda_i\delta_i) & -\lambda_i \sin(\lambda_i\delta_i) & \lambda_i \cos(\lambda_i\delta_i) \\ \lambda_i^2 \cosh(\lambda_i\delta_i) & \lambda_i^2 \sinh(\lambda_i\delta_i) & -\lambda_i^2 \cos(\lambda_i\delta_i) & -\lambda_i^2 \sin(\lambda_i\delta_i) \\ \lambda_i^3 \sinh(\lambda_i\delta_i) & \lambda_i^3 \cosh(\lambda_i\delta_i) & \lambda_i^3 \sin(\lambda_i\delta_i) & -\lambda_i^3 \cos(\lambda_i\delta_i) \end{bmatrix} \qquad \delta_i = e_i - e_{i-1} \tag{12}$$

2.2.2 Concentrated mass

Now suppose that ith discontinuity is a concentrated mass. Compatibility conditions can be considered as continuity of vertical displacement and, slope and, discontinuity of bending moment, and shear force. Using Eq.(8), compatibility conditions can be rewritten as:

$$v_i = v_{i+1} \quad v_i' = v_{i+1}' \quad v_{i+1}'' - v_i'' = -J_i\, \lambda_i^4\, v_{i+1}' \quad v_{i+1}''' - v_i''' = M_i\, \lambda_i^4\, v_{i+1} , \tag{13}$$

where

$$M_i = \frac{m_i}{\rho A_i L} \quad J_i = \frac{j_i}{\rho A_i L^3} = M_i c_i^2 . \tag{14}$$

It's obvious that $\lambda_i = \lambda_{i+1}$ when ith discontinuity is a concentrated mass. In a similar manner, one can derive Eqs. (11) and (12), where

$$P^{(i)} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & J_i\lambda_i^3 & -1 & J_i\lambda_i^3 \\ -M_i\lambda_i & 1 & -M_i\lambda_i & -1 \end{bmatrix} \qquad Q^{(i)} = \begin{bmatrix} \cosh(\lambda_i\delta_i) & \sinh(\lambda_i\delta_i) & \cos(\lambda_i\delta_i) & \sin(\lambda_i\delta_i) \\ \sinh(\lambda_i\delta_i) & \cosh(\lambda_i\delta_i) & -\sin(\lambda_i\delta_i) & \cos(\lambda_i\delta_i) \\ \cosh(\lambda_i\delta_i) & \sinh(\lambda_i\delta_i) & -\cos(\lambda_i\delta_i) & -\sin(\lambda_i\delta_i) \\ \sinh(\lambda_i\delta_i) & \cosh(\lambda_i\delta_i) & \sin(\lambda_i\delta_i) & -\cos(\lambda_i\delta_i) \end{bmatrix}. \tag{15}$$

Table 1
Mathematical model for boundary conditions in Bernoulli-Euler beam theory.

Boundary conditions	Mathematical model(α =1-e_n)	
Simply supported(SS)	$v_1(0) = v_1''(0) = 0$	$v_{n+1}(\alpha) = v_{n+1}''(\alpha) = 0$
Simple–Clamped(SC)	$v_1(0) = v_1''(0) = 0$	$v_{n+1}(\alpha) = v_{n+1}'(\alpha) = 0$
Clamped- Clamped(CC)	$v_1(0) = v_1'(0) = 0$	$v_{n+1}(\alpha) = v_{n+1}'(\alpha) = 0$
Cantilever(CF)	$v_1(0) = v_1'(0) = 0$	$v_{n+1}''(\alpha) = v_{n+1}'''(\alpha) = 0$

Table 2
Definition of matrix $[\Lambda]$ for various boundary conditions in Bernoulli-Euler beam theory.

Boundary conditions of the right side of the beam	$[\Lambda]$
Simply supported	$\begin{bmatrix} \cosh(\lambda_{n+1}\alpha) & \sinh(\lambda_{n+1}\alpha) & \cos(\lambda_{n+1}\alpha) & \sin(\lambda_{n+1}\alpha) \\ \cosh(\lambda_{n+1}\alpha) & \sinh(\lambda_{n+1}\alpha) & -\cos(\lambda_{n+1}\alpha) & -\sin(\lambda_{n+1}\alpha) \end{bmatrix}$
Clamped	$\begin{bmatrix} \cosh(\lambda_{n+1}\alpha) & \sinh(\lambda_{n+1}\alpha) & \cos(\lambda_{n+1}\alpha) & \sin(\lambda_{n+1}\alpha) \\ \sinh(\lambda_{n+1}\alpha) & \cosh(\lambda_{n+1}\alpha) & -\sin(\lambda_{n+1}\alpha) & \cos(\lambda_{n+1}\alpha) \end{bmatrix}$
Free	$\begin{bmatrix} \cosh(\lambda_{n+1}\alpha) & \sinh(\lambda_{n+1}\alpha) & -\cos(\lambda_{n+1}\alpha) & -\sin(\lambda_{n+1}\alpha) \\ \sinh(\lambda_{n+1}\alpha) & \cosh(\lambda_{n+1}\alpha) & \sin(\lambda_{n+1}\alpha) & -\cos(\lambda_{n+1}\alpha) \end{bmatrix}$

2.3 Implementation of external boundary conditions

In what follows, four common boundary conditions are considered to derive frequency equation. Mathematical model of these boundary conditions is presented in Table 1. Boundary conditions at the right side of the beam (ζ=1) can be written in a matrix form as:

$$[\Lambda]\{A_{n+1} \quad B_{n+1} \quad C_{n+1} \quad D_{n+1}\}^T = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{16}$$

In which definition of the matrix $[\Lambda]$ is presented in Table 2. for various boundary cases.

By substitution of Eq. (11) into the Eq. (16) for i=n, n-1,…,2,1, next relation appears as:

$$[\Gamma]\{A_1 \quad B_1 \quad C_1 \quad D_1\}^T = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{17}$$

where

$$\Gamma = \Lambda T^{(n)} T^{(n-1)} \dots T^{(2)} T^{(1)}. \tag{18}$$

For beams which their left side (ζ=0) is simply supported, implementation of boundary conditions at this side leads to $A_1=C_1=0$; therefore one can simplify Eq. (17) as:

$$\begin{bmatrix} \Gamma_{12} & \Gamma_{14} \\ \Gamma_{22} & \Gamma_{24} \end{bmatrix} \begin{Bmatrix} B_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{19}$$

But, when left side of the beam is clamped, implementation of external boundary conditions at the left side of the beam leads to $A_1+C_1=0$ and $B_1+D_1=0$; hence, frequency equation can be derived as:

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{20}$$

In order to find non-trivial solutions of Eqs. (19) or (20), the determinant of coefficient matrices should be put equal to zero.

3 TIMOSHENKO BEAM THEORY

As Timoshenko beam theory considers transverse deformation and rotational inertia of the beam, this model is much accurate than Bernoulli-Euler one, especially for the short beams. This model is investigated here as it was presented in previous section for Bernoulli-Euler beam.

3.1 Governing equations

The governing set of equations of free bending vibration of a bare Timoshenko beam are written as [23]

$$\begin{aligned} &-\frac{\partial}{\partial x}\left\{kGA(x)\left[\psi(x,t)-\frac{\partial w(x,t)}{\partial x}\right]\right\}=\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}, \\ &-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial \psi(x,t)}{\partial x}\right]+kGA(x)\left[\psi(x,t)-\frac{\partial w(x,t)}{\partial x}\right]=\rho I(x)\frac{\partial^2 \psi(x,t)}{\partial t^2} \end{aligned} \tag{21}$$

where ψ (x,t) is the rotation due to bending, *G* is the shear modulus of material and *k* is called "shear correction factor" introduced to make up the geometry-dependent distribution of shear stress. This factor depends on the shape of section and Poisson's ratio of material [24].

For *i*th segment of the beam, Eq. (21) can be rewritten as:

$$\begin{aligned} &kG\frac{\partial}{\partial x}\left[\psi_i(x,t)-\frac{\partial w_i(x,t)}{\partial x}\right]+\rho\frac{\partial^2 w_i(x,t)}{\partial t^2}=0 \\ &EI_i\frac{\partial^2 \psi_i(x,t)}{\partial x^2}-kGA_i\left[\psi_i(x,t)-\frac{\partial w_i(x,t)}{\partial x}\right]+\rho I_i\frac{\partial^2 \psi_i(x,t)}{\partial t^2}=0 \qquad i=1,2,\ldots,n+1. \end{aligned} \tag{22}$$

The vertical displacement w_i(x,t) and rotation due to the bending ψ_i (x,t) can be assumed as the product of the functions v_i(x) and ψ_i (x) depended on the spatial coordinate *x* and *a* time dependent harmonic function as:

$$w_i(x,t)=Lv_i(x)e^{j\omega t} \quad \psi_i(x,t)=\Psi_i(x)e^{j\omega t}. \tag{23}$$

Substitution of the separated form provided by Eq. (23) into the governing Eqs. (22) and also using Eq. (5) and new dimensionless parameters defined as:

$$r_i^2=\frac{I_i}{A_iL^2} \quad s_i^2=\frac{EI_i}{kGA_iL^2}=\frac{2(1+\upsilon)}{k}r_i^2, \tag{24}$$

yields the following set of differential equations:

$$v_i'' - \Psi_i' + \lambda_i^4 s_i^2 v_i = 0 \quad s_i^2 \Psi'' + v_i' - (1 - \lambda_i^4 s_i^2 r_i^2)\Psi_i = (\tag{25}$$

Using the new variables given by

$$\begin{gathered} m_{1i} = \frac{\beta_{1i}^2 + \lambda_i^4 s_i^2}{\beta_{1i}} \quad \beta_{1i} = \sqrt{-d_1 + \sqrt{d_1^2 - d_{2i}}} \quad d_1 = \frac{\lambda_1^4 (r_1^2 + s_1^2)}{2} \\ m_{2i} = \frac{-\beta_{2i}^2 + \lambda_i^4 s_i^2}{\beta_{2i}} \quad \beta_{2i} = \sqrt{d_1 + \sqrt{d_1^2 - d_{2i}}} \quad d_{2i} = \lambda_i^4 (\lambda_i^4 s_i^2 r_i^2 - 1) \end{gathered} \tag{26}$$

Solution of the set of Eqs. (25) can be found as:

$$\begin{gathered} v_i(\zeta) = A_i \cosh(\beta_{1i}\zeta) + B_i \sinh(\beta_{1i}\zeta) + C_i \cos(\beta_{2i}\zeta) + D_i \sin(\beta_{2i}\zeta) \\ \Psi_i(\zeta) = A_i m_{1i} \sinh(\beta_{1i}\zeta) + B_i m_{1i} \cosh(\beta_{1i}\zeta) + C_i m_{2i} \sin(\beta_{2i}\zeta) - D_i m_{2i} \cos(\beta_{2i}\zeta) \end{gathered} \tag{27}$$

3.2 Compatibility conditions

In the vicinity of *i*th discontinuity, Eq. (27) can be written as:

$$\begin{aligned} & v_i(\zeta - e_{i-1}) = A_i \cosh\left[\beta_{1i}(\zeta - e_{i-1})\right] + B_i \sinh\left[\beta_{1i}(\zeta - e_{i-1})\right] + C_i \cos\left[\beta_{2i}(\zeta - e_{i-1})\right] + \\ & D_i \sin\left[\beta_{2i}(\zeta - e_{i-1})\right] \\ & \Psi_i(\zeta - e_{i-1}) = A_i m_{1i} \sinh\left[\beta_{1i}(\zeta - e_{i-1})\right] + B_i m_{1i} \cosh\left[\beta_{1i}(\zeta - e_{i-1})\right] + C_i m_{2i} \sin\left[\beta_{2i}(\zeta - e_{i-1})\right] - \\ & D_i m_{2i} \cos\left[\beta_{2i}(\zeta - e_{i-1})\right] \\ & v_{i+1}(\zeta - e_i) = A_{i+1} \cosh\left[\beta_{1(i+1)}(\zeta - e_i)\right] + B_{i+1} \sinh\left[\beta_{1(i+1)}(\zeta - e_i)\right] + C_{i+1} \cos\left[\beta_{2(i+1)}(\zeta - e_i)\right] + \\ & D_{i+1} \sin\left[\beta_{2(i+1)}(\zeta - e_i)\right] \\ & \Psi_{i+1}(\zeta - e_i) = A_{i+1} m_{1(i+1)} \sinh\left[\beta_{1(i+1)}(\zeta - e_i)\right] + B_{i+1} m_{1(i+1)} \cosh\left[\beta_{1(i+1)}(\zeta - e_i)\right] + \\ & C_{i+1} m_{2(i+1)} \sin\left[\beta_{2(i+1)}(\zeta - e_i)\right] - D_{i+1} m_{2(i+1)} \cos\left[\beta_{2(i+1)}(\zeta - e_i)\right] \end{aligned} \tag{28}$$

3.2.1 Stepped section

According to the definitions of the natural parameters in Timoshenko beam theory as [23]

$$M_i = \frac{EI_i}{L} \Psi_i' \quad V_i = kA_i G\left(\Psi_i - v_i'\right), \tag{29}$$

The compatibility conditions at the stepped section can be written as:

$$v_i = v_{i+1} \quad \Psi_i = \Psi_{i+1} \quad \Psi_i' = \xi_i \Psi_{i+1}' \quad \Psi_i - v_i' = \mu_i\left(\Psi_{i+1} - v_{i+1}'\right), \tag{30}$$

where

$$\mu_i = \frac{A_{i+1}}{A_i}. \tag{31}$$

Using Eqs. (28) and (30), one can obtain Eqs. (11) - (12), where

$$P^{(i)}=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & m_{1(i+1)} & 0 & -m_{2(i+1)} \\ m_{1(i+1)}\beta_{1(i+1)}\xi_i & 0 & m_{2(i+1)}\beta_{2(i+1)}\xi_i & 0 \\ 0 & \left(m_{1(i+1)}-\beta_{1(i+1)}\right)\mu_i & 0 & -\left(m_{2(i+1)}+\beta_{2(i+1)}\right)\mu_i\xi_i \end{bmatrix} \tag{32}$$

$$Q^{(i)}=\begin{bmatrix} \cosh(\beta_{1i}\delta_i) & \sinh(\beta_{1i}\delta_i) & \cos(\beta_{2i}\delta_i) & \sin(\beta_{2i}\delta_i) \\ m_{1i}\sinh(\beta_{1i}\delta_i) & m_{1i}\cosh(\beta_{1i}\delta_i) & m_{2i}\sin(\beta_{2i}\delta_i) & -m_{2i}\cos(\beta_{2i}\delta_i) \\ m_{1i}\beta_{1i}\cosh(\beta_{1i}\delta_i) & m_{1i}\beta_{1i}\sinh(\beta_{1i}\delta_i) & m_{2i}\beta_{2i}\cos(\beta_{2i}\delta_i) & m_{2i}\beta_{2i}\sin(\beta_{2i}\delta_i) \\ (m_{1i}-\beta_{1i})\sinh(\beta_{1i}\delta_i) & (m_{1i}-\beta_{1i})\cosh(\beta_{1i}\delta_i) & (m_{2i}+\beta_{2i})\sin(\beta_{1i}\delta_i) & -(m_{2i}+\beta_{2i})\cos(\beta_{1i}\delta_i) \end{bmatrix}$$

3.2.2 Concentrated mass

Using Eq. (29), compatibility conditions at the location of the attached mass can be written as:

$$v_i=v_{i+1} \quad \Psi_i=\Psi_{i+1} \quad v_{i+1}'-v_i'=-M_i s_i^2\lambda_i^4 v_{i+1} \quad \Psi_{i+1}'-\Psi_i'=-J_i\lambda_i^4\Psi_{i+1}. \tag{33}$$

It's obvious that $\lambda_i=\lambda_{i+1}, r_i=r_{i+1}$ and $s_i=s_{i+1}$ when *i*th discontinuity is a concentrated mass. In a similar manner, one can obtain Eqs. (11) – (12), where

$$P^{(i)}=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & m_{1i} & 0 & -m_{2i} \\ M_i s_i^2\lambda_i^4 & \beta_{1i} & M_i s_i^2\lambda_i^4 & \beta_{2i} \\ m_{1i}\beta_{1i} & J_i m_{1i}\lambda_i^4 & m_{2i}\beta_{2i} & -J_i m_{2i}\lambda_i^4 \end{bmatrix} \tag{34}$$

$$Q^{(i)}=\begin{bmatrix} \cosh(\beta_{1i}\delta_i) & \sinh(\beta_{1i}\delta_i) & \cos(\beta_{2i}\delta_i) & \sin(\beta_{2i}\delta_i) \\ m_{1i}\sinh(\beta_{1i}\delta_i) & m_{1i}\cosh(\beta_{1i}\delta_i) & m_{2i}\sin(\beta_{2i}\delta_i) & -m_{2i}\cos(\beta_{2i}\delta_i) \\ \beta_{1i}\sinh(\beta_{1i}\delta_i) & \beta_{1i}\cosh(\beta_{1i}\delta_i) & -\beta_{2i}\sin(\beta_{2i}\delta_i) & \beta_{2i}\cos(\beta_{2i}\delta_i) \\ m_{1i}\beta_{1i}\cosh(\beta_{1i}\delta_i) & m_{1i}\beta_{1i}\sinh(\beta_{1i}\delta_i) & m_{2i}\beta_{2i}\cos(\beta_{1i}\delta_i) & m_{2i}\beta_{2i}\sin(\beta_{1i}\delta_i) \end{bmatrix}.$$

3.3 Implementation of external boundary conditions

In Timoshenko beam theory, mathematical model of the above mentioned boundary conditions is presented in Table 3.

Table 3
Mathematical model for external boundary conditions in Timoshenko beam theory.

Boundary conditions	Mathematical model(α =1-e$_n$)	
Simply supported(SS)	$v_1(0)=\Psi_1'(0)=0$	$v_{n+1}(\alpha)=\Psi_{n+1}'(\alpha)=0$
Simple–Clamped(SC)	$v_1(0)=\Psi_1'(0)=0$	$v_{n+1}(\alpha)=\Psi_{n+1}(\alpha)=0$
Clamped-Clamped(CC)	$v_1(0)=\Psi_1(0)=0$	$v_{n+1}(\alpha)=\Psi_{n+1}(\alpha)=0$
Cantilever(CF)	$v_1(0)=\Psi_1(0)=0$	$\Psi_{n+1}'(\alpha)=\left(\Psi_{n+1}-v_{n+1}'\right)(\alpha)=0$

Table 4
Definition of matrix [Λ] for various boundary conditions in Timoshenko beam theory.

Boundary conditions of the right side of the beam	[Λ]
Simply supported	$\begin{bmatrix} \cosh(\beta_{1(n+1)}\alpha) & m_{1(n+1)}\beta_{1(n+1)}\cosh(\beta_{1(n+1)}\alpha) \\ \sinh(\beta_{1(n+1)}\alpha) & m_{1(n+1)}\beta_{1(n+1)}\sinh(\beta_{1(n+1)}\alpha) \\ \cos(\beta_{2(n+1)}\alpha) & m_{2(n+1)}\beta_{2(n+1)}\cos(\beta_{2(n+1)}\alpha) \\ \sin(\beta_{2(n+1)}\alpha) & m_{2(n+1)}\beta_{2(n+1)}\sin(\beta_{2(n+1)}\alpha) \end{bmatrix}^T$
Clamped	$\begin{bmatrix} \cosh(\beta_{1(n+1)}\alpha) & \mathrm{m}_{1(n+1)}\sinh(\beta_{1(n+1)}\alpha) \\ \sinh(\beta_{1(n+1)}\alpha) & \mathrm{m}_{1(n+1)}\cosh(\beta_{1(n+1)}\alpha) \\ \cos(\beta_{2(n+1)}\alpha) & \mathrm{m}_{2(n+1)}\sin(\beta_{2(n+1)}\alpha) \\ \sin(\beta_{2(n+1)}\alpha) & -\mathrm{m}_{2(n+1)}\cos(\beta_{2(n+1)}\alpha) \end{bmatrix}^T$
Free	$\begin{bmatrix} m_{1(n+1)}\beta_{1(n+1)}\cosh(\beta_{1(n+1)}\alpha) & (\mathrm{m}_{1(n+1)}-\beta_{1(n+1)})\sinh(\beta_{1(n+1)}\alpha) \\ m_{1(n+1)}\beta_{1(n+1)}\sinh(\beta_{1(n+1)}\alpha) & (\mathrm{m}_{1(n+1)}-\beta_{1(n+1)})\cosh(\beta_{1(n+1)}\alpha) \\ m_{2(n+1)}\beta_{2(n+1)}\cos(\beta_{2(n+1)}\alpha) & (\mathrm{m}_{2(n+1)}+\beta_{2(n+1)})\sin(\beta_{2(n+1)}\alpha) \\ m_{2(n+1)}\beta_{2(n+1)}\sin(\beta_{2(n+1)}\alpha) & -(\mathrm{m}_{2(n+1)}+\beta_{2(n+1)})\cos(\beta_{2(n+1)}\alpha) \end{bmatrix}^T$

In a similar manner, for beams which their left side is simply supported, frequency equation can be obtained as:

$$\begin{bmatrix} \Gamma_{12} & \Gamma_{14} \\ \Gamma_{22} & \Gamma_{24} \end{bmatrix} \begin{Bmatrix} B_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{35}$$

and for beams which their left side is clamped following equation can be derived:

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ 1 & 0 & 1 & 0 \\ 0 & \mathrm{m}_1 & 0 & -\mathrm{m}_2 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{36}$$

where definition of matrix [Γ] is presented in Eq. (18) and matrix [Λ] is presented in Table 4. for various boundary conditions.

4 DERIVATION OF MODE SHAPES

For both the Bernoulli-Euler and Timoshenko beam theories, using obtained eigenvalues, one can evaluate eigenvectors using Eqs. (7) or (28), (11) and (19), (20), (35) or (36) and calculate mode shapes using Heaviside function as follow:

$$v(\zeta) = v_1(\zeta) + \sum_{i=1}^{n} \left[v_{i+1}(\zeta) - v_i(\zeta) \right] \mathrm{H}(\zeta - \zeta_i). \tag{37}$$

Finally it should be stated than in order to normalize mode shapes, each mode will be normalized as:

$$\bar{v} = \frac{v}{\max(|v|)}. \tag{38}$$

5 NUMERICAL RESULTS AND DISCUSSIONS

In what follows, the numerical results of the solution of the frequency equation are presented and discussed for various cases.

Consider a stepped cantilever Bernoulli-Euler beam with three steps at ζ =0.25, 0.55 and 0.8, constant thickness and variable height as h_2/h_1=0.8, h_3/h_1=0.65 and h_4/h_1=0.25. First four dimensionless frequencies are calculated and tabulated in Table 5 and corresponding normal modes are illustrated in Fig. 2. A comparison between the results and the ones proposed by Mao [22], confirms the accuracy of the proposed method.

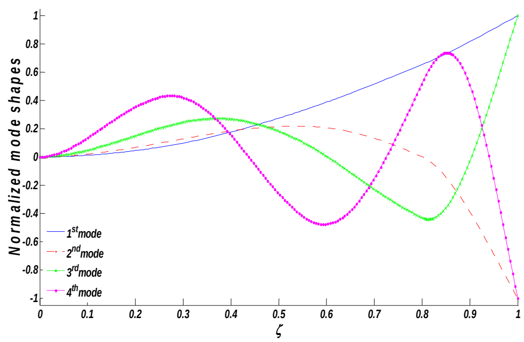


Fig. 2
First four normalized modes of three stepped cantilever Bernoulli-Euler beam.

Table 5
First four dimensionless frequencies of three stepped cantilever Bernoulli-Euler beam.

	λ_1 (1)	λ_1 (2)	λ_1 (3)	λ_1 (4)
present	2.1786	4.2354	5.9213	8.4620
Mao [22]	2.1785	4.2357	5.9220	8.4620

Table 6
First three dimensionless frequency of a simply supported Bernoulli-Euler beam with two symmetric concentrated masses.

	c_1=c_2=0.01		c_1=c_2=0.1	
	present	Maiz et al. [14]	present	Maiz et al. [14]
λ_1 (1)	3.0010	3.0012	2.9888	2.9892
λ_1 (2)	5.7742	5.7745	5.7745	5.7745
λ_1 (3)	9.0555	9.0559	8.6819	8.6820

Table 7
Values of [λ_1^2 (1)- λ_1^2 (3)] for a cantilever Timoshenko beam with a tip mass.

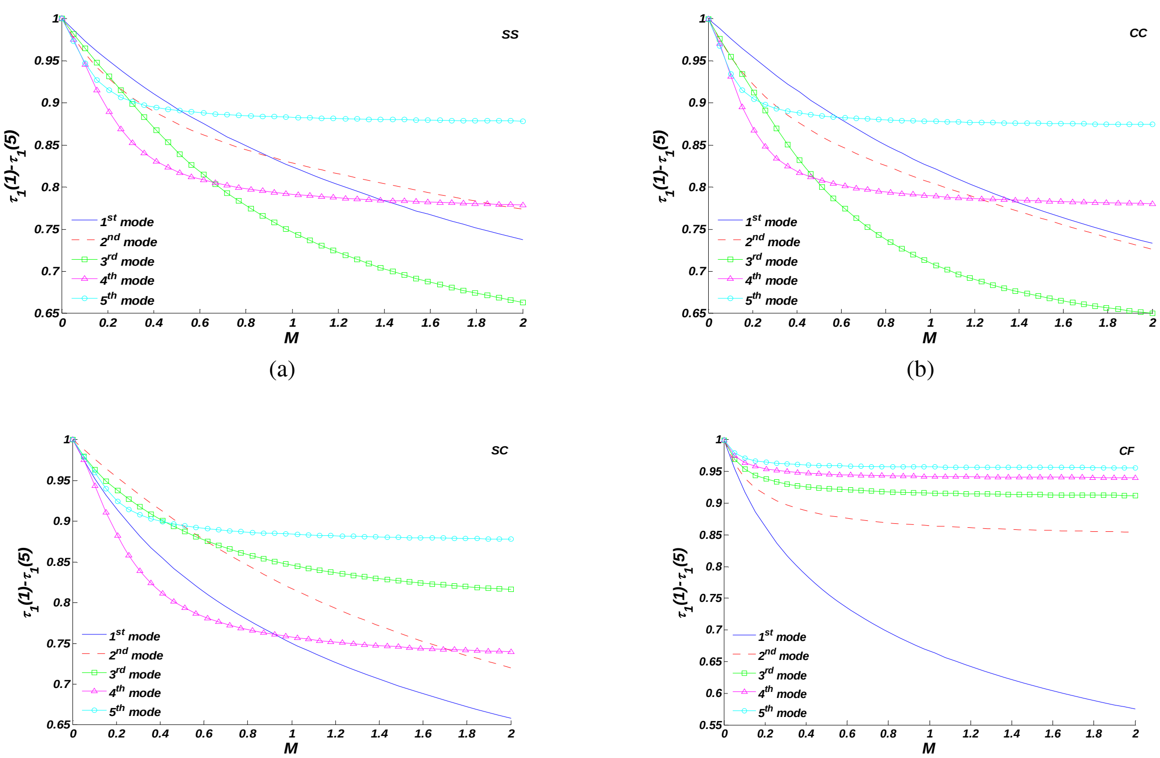
	c	M=0.2			M=0.4			M=0.6		
present	0	2.5666	16.1755	41.6629	2.1343	15.3349	40.6289	1.8645	14.9380	40.1851
Rossi and Laura [12]		2.567	16.177	41.673	2.135	15.335	40.632	1.865	14.940	40.190
present	0.1	2.5556	15.4376	37.7985	2.1208	14.1701	34.0834	1.8512	13.4052	31.4396
Rossi and Laura [12]		2.556	15.438	37.804	2.121	14.172	34.094	1.852	13.406	31.441

Now consider a uniform simply supported Bernoulli-Euler beam carrying two symmetric concentrated masses (M_1=M_2=0.1) located at ζ =0.25 and 0.75 with two values of rotary inertia (c_1=c_2=0.01 and 0.1). In Table 6. , the first three frequencies are presented and are compared with exact closed-form solution proposed by Maiz et al. [14]. This comparison confirms the versatility of the proposed solution.

A cantilever Timoshenko beam with a tip mass and properties have been mentioned in [12] is considered, Table 7. shows square value of the first three frequencies for various amounts of mass and rotary inertia. Comparison of the results reveals the high accuracy of presented solution.

After validation of the proposed solution, effect of the various parameters on the frequencies can be investigated for all the boundary conditions. In what follows, all results are derived for a Timoshenko beam with dimensionless parameters as r=0.03 and s=0.05. Also, in order to be able to show all frequencies simultaneously, each frequency is divided to the corresponding value of a uniform one without any attachment (Ω), in other words $T_1(j)=\lambda_1(j)/\Omega(j)$.

To study the effect of the value of translational inertia on the natural frequencies, consider simply supported and clamped-clamped beams with a concentrated mass located at ζ =0.25, one simple-clamped beam with a concentrated mass located at the middle section and a cantilever one with a tip mass; Fig. 3 shows the effect of value of mass on the first five frequencies. As shown for all boundary conditions, when the value of mass increases, all frequencies decrease.



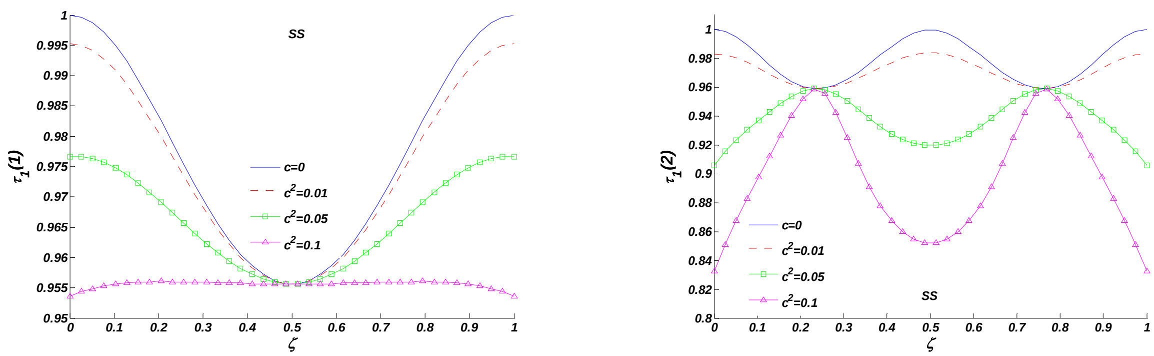
(a) (b)

(c) (d)

Fig. 3
First five frequency of simply supported and clamped-clamped beams with a concentrated mass located at ζ =0.25, a simple-clamped beam with a concentrated mass located at middle section and a cantilever one with a tip mass.

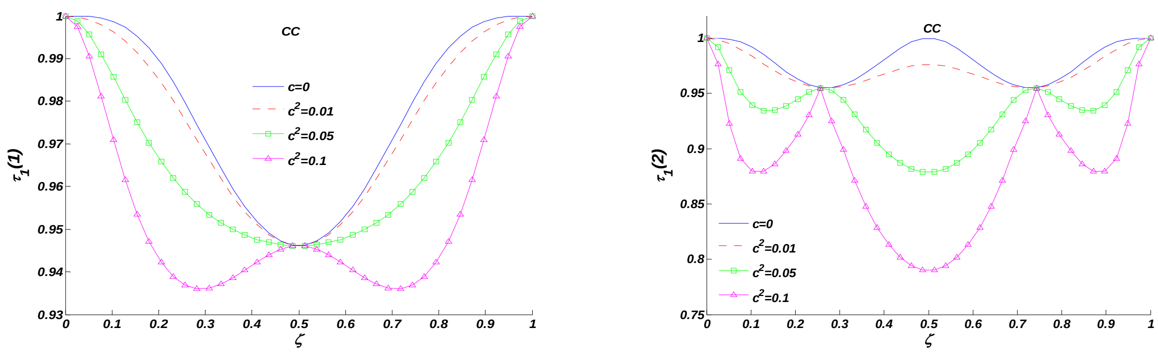
Now, a uniform beam with a concentrated mass (M=0.1) is considered. Value of the first two frequencies are depicted versus the position of the mass for variable values of rotary inertia for different boundary conditions in Figs. 4-7. As shown in these figures, when value of rotary inertia increases, value of the frequencies decreases. Figs. 4-7 also show that in each mode, there are some points that when mass is located on them, there is no decrease in the frequencies for c=0; on the other hand for high value of inertia, maximum decrease in frequencies happens when the mass is located at these points. In other words, when mass located at these points, all decrease in corresponding frequency is effected by rotary inertia whereas translational inertia has no effect on corresponding frequency. These points are the nodes in corresponding mode, e.g. center point for even frequencies of symmetric beams. Also, there are some points that when the mass is located on them, value of decrease in frequency is independent from rotary inertia. in other words, when the mass located at these points, all decreases in corresponding frequency is effected by translational inertia whereas rotary inertia has no effect on the corresponding frequency. These points are antinodes of corresponding mode shape, e. g. center point for odd frequencies of symmetric beams.

To investigate stepped beams with concentrated masses, consider a simply supported shaft. The value of diameter decreases to half in ζ =0.8, with a gear (M=0.1) located before step; Fig. 8 shows the effect of the position of the gear and the value of its rotary inertia on the first four frequencies. Similar results with the case of uniform beam can be obtained.



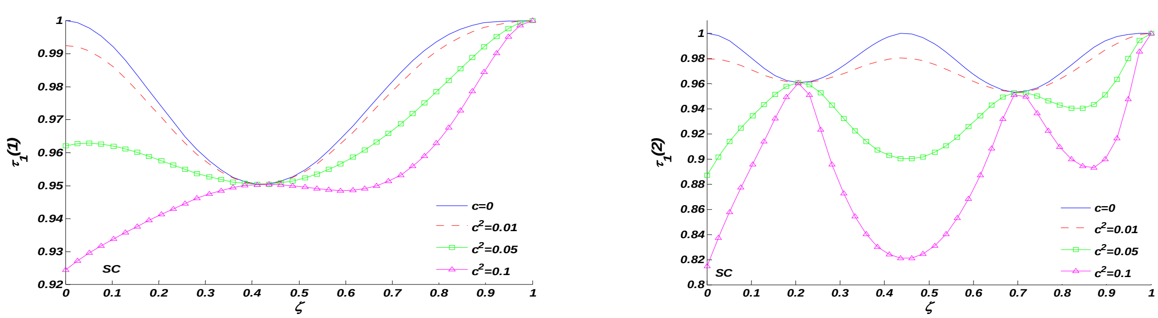
(a) (b)

Fig. 4
First two frequency versus position of mass for variable values of rotary inertia, for a simply supported beam.



(a) (b)

Fig. 5
First two frequency versus position of mass for variable values of rotary inertia, for a clamped-clamped beam.



(a) (b)

Fig.6
First two frequency versus position of mass for variable values of rotary inertia, for a simple-clamped beam.

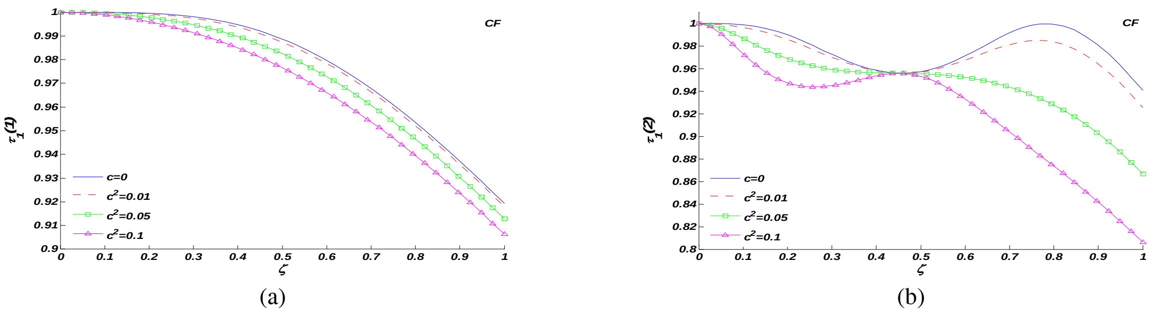


Fig. 7
First two frequency versus position of mass for variable values of rotary inertia , for a cantilever beam.

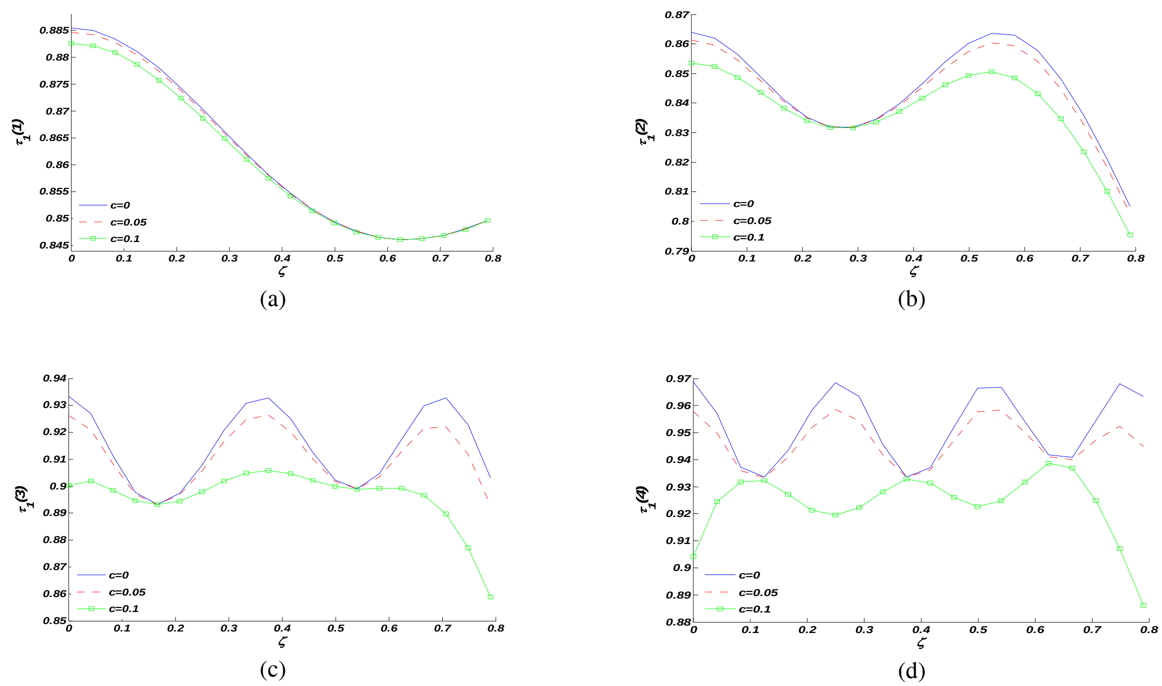


Fig. 8
First four frequency of simply supported stepped beam with a concentrated mass having rotary inertia.

6 CONCLUSIONS

Using transfer matrix method, vibration analysis of multi-step beam carrying concentrated masses was presented analytically. Both Bernoulli-Euler and Timoshenko beam theories was used to model a beam mathematically. Generally, in analyzing the vibration of the beams carrying attached masses, only the translational inertia of the mass is considered. In those cases, it is generally observed that natural frequencies decrease with respect to the values of the mass, except for the cases in which the masses are located at nodal points of the corresponding normal mode. On the other hand, when the model takes into account the rotary inertia of the mass too, all the natural frequencies of vibration decrease for all cases. The effect of the translational inertia has its highest influence over a natural frequency, when the mass is located at an antinode of the corresponding normal mode. In that situation the rotary inertia has no effect. The effect of the rotary inertia has its highest influence over a natural frequency when the mass is located at a node of the normal mode. In that situation the translational inertia has no effect. These results are valid for both uniform and multiple-stepped beams.

REFERENCES

[1] Chen Y., 1963, On the vibration of beams or rods carrying a concentrated mass, *Journal of Applied Mechanics* **30**:310-311.

[2] Laura P., Maurizi M.J., Pombo J.L., 1975, A note on the dynamics analysis of an elastically restrained-free beam with a mass at the free end, *Journal of Sound and Vibration* **41**:397-405.

[3] Laura P., Verniere de Irassar P.L., Ficcadenti G.M, 1983, A note on transverse vibration of continuous beams subjected to an axial force and carrying concentrated masses, *Journal of Sound and Vibration* **86**:279-284.

[4] Gurgoze M., 1984, A note on the vibrations of restrained beams and rods with point masses, *Journal of Sound and Vibration* **96**:461-468.

[5] Gurgoze M., 1985, On the vibration of restrained beams and rods with heavy masses, *Journal of Sound and Vibration* **100**:588-589.

[6] Liu W.H., Wu J.R., Huang C.C., 1988, Free vibrations of beams with elastically restrained edges and intermediate concentrated masses, *Journal of Sound and Vibration* **122**:193-207.

[7] De Rosa M.A., Franciosi C., Maurizi M.J., 1955, On the dynamics behaviour of slender beams with elastic ends carrying a concentrated mass, *Computers and Structures* **58**:1145-1159.

[8] Rossit C.A., Laura P., 2001, Transverse vibrations of a cantilever beam with a spring mass system attached on the free end, *Ocean Engineering* **28**:933-939.

[9] Rao G.V., Saheb K.M., Janardhan G.R., 2006, Fundamental frequency for large amplitude vibrations of uniform Timoshenko beams with central point concentrated mass using coupled displacement field method, *Journal of Sound and Vibration* **298**:221-232.

[10] Rossit M.C.A., Laura P., 2001, Transverse normal modes of vibration of a cantilever Timoshenko beam with a mass elastically mounted at the free end, *Journal of the Acoustical Society of America* **110**:2837-2840.

[11] Laura P., Filipich C.P., Cortinez V.H., 1987, Vibrations of beams and plates carrying concentrated masses, *Journal of Sound and Vibration* **117**:459-465.

[12] Rossi R.E., Laura P, 1990, Vibrations of a Timoshenko beam clamped at one end and carrying a finite mass at the other, *Applied Acoustics* **30**:293-301.

[13] Chang C.H., 2000, Free vibration of a simply supported beam carrying a rigid mass at the middle, *Journal of Sound and Vibration* **237**:4733-4744.

[14] Maiz S., Bambill D., Rossit C., Laura P., 2007, Transverse vibration of Bernoulli–Euler beams carrying point masses and taking into account their rotary inertia, *Journal of Sound and Vibration* **303**:895-908.

[15] Lin H.Y., 2009, On the natural frequencies and mode shapes of a multi-span Timoshenko beam carrying a number of various concentrated elements, *Journal of Sound and Vibration* **319**:593-605.

[16] Demirdag O., Murat Y.S., 2009, Free vibration analysis of elastically supported Timoshenko columns with attached masses using fuzzy neural network, *Journal of Scientific and Industrial Researches* **68**:285-291.

[17] Demirdag O., Yesilce Y., 2011, Solution of free vibration equation of elastically supported Timoshenko columns with a tip mass by differential transform method, *Advances in Engineering Software* **42**:860-867.

[18] Gutierrez R.H., Laura P., Rossi R.E., 1991, Vibrations of a Timoshenko beam of non-uniform cross-section elastically restrained at one end and carrying a finite mass at the other, *Ocean Engineering* **18**:129-145.

[19] Naguleswaran S., 2002, Vibration of an Euler–Bernoulli beam on elastic end supports and with up to three step changes in cross-section, *International Journal of Mechanical Science* **44**:2541-2555.

[20] Kukla Y., Zamojska I., 2007, Frequency analysis of axially loaded stepped beams by Green's function method, *Journal of Sound and Vibration* **300**:1034-1041.

[21] Lu Z.R., Huang M., Liu J.K., Chen W.H., Liao W.Y., 2009, Vibration analysis of multiple-stepped beams with the composite element model, *Journal of Sound and Vibration* **322**:1070-1080.

[22] Mao Q., 2011, Free vibration analysis of multiple-stepped beams by using Adomian decomposition method, *Mathematical and Computer Modelling* **54**:756-764.

[23] Timoshenko S., Weaver W., Young D.H., 1974, *Vibration Problems in Engineering*, Wiley, New York.

[24] Hutchinson J.R., 2001, Shear coefficients for Timoshenko beam theory, *ASME Journal of Applied Mechanics* **68**:87-92.