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Fractional Order Generalized Thermoelastic Functionally Graded Solid with Variable Material Properties

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ABSTRACT

In this work, a new mathematical model of thermoelasticity theory has been considered in the context of a new consideration of heat conduction with fractional order theory. A functionally graded isotropic unbounded medium is considered subjected to a periodically varying heat source in the context of space-time non-local generalization of three-phase-lag thermoelastic model and Green-Naghdi models, in which the thermophysical properties are temperature dependent. The governing equations are expressed in Laplace-Fourier double transform domain and solved in that domain. Then the inversion of the Fourier transform is carried out by using residual calculus, where poles of the integrand are obtained numerically in complex domain by using Laguerre's method and the inversion of Laplace transform is done numerically using a method based on Fourier series expansion technique. The numerical estimates of the thermal displacement, temperature and thermal stress are obtained for a hypothetical material. Finally, the obtained results are presented graphically to show the effect of non-local fractional parameter on thermal displacement, temperature and thermal stress. A comparison of the results for different theories (three-phase-lag model, GN model II, GN model III) is presented and the effect of non-homogeneity is also shown. The results, corresponding to the cases, when the material properties are temperature independent, agree with the results of the existing literature.

Keywords: Three-phase-lag thermoelastic model; Temperature dependent elastic parameters; Fractional order heat equations; Periodically varying heat source; Functionally graded materials

1 INTRODUCTION

STUDYING the modification of heat conduction equation from diffusive to a wave type may be affected either by a microscopic consideration of the phenomenon of heat transport or in a phenomenological way by modifying the classical Fourier law of heat conduction. The first is due to Cattaneo [1], who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier law. Puri and Kythe [2] investigated the effects of using the Maxwell-Cattaneo model in Stock's problem for a viscous fluid and they note that, the non-dimensional thermal relaxation time defined as to CP_r, where C and P_r are the Cattaneo and Prandtl number, respectively, is of order 10^{-2}.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is

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well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic, and this is one reason why fractional calculus has become more and more popular [3–5].

Although the tools of fractional calculus were available and applicable to various fields of study, the investigation into the theory of fractional differential equation started quite recently [3]. The differential equations involving Riemann-Liouville differential operators of fractional order $0 < \alpha < 1$ appear to be important in modeling several physical phenomena [6] and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.

Recently, a considerable research effort is expended to study anomalous diffusion, which is characterized by the time-fractional diffusion-wave equation by Kimmich [8] as follows:

$$\rho c = \lambda I^{\xi} \nabla^2 c, \quad 0 \; b \; \xi \; b \; 2 \tag{1}$$

where ρ is the mass density, c is the concentration, λ is the diffusion conductivity and the notation I^{ξ} is the Riemann-Liouville fractional integral, introduced as a natural generalization of the well-known $\xi-$fold repeated integral written in a convolution-type form.

It should be noted that the term diffusion is often used in a more generalized sense including various transport phenomena. Eq. (1) is a mathematical model of a wide range of important physical phenomena [9–12], for example, the sub-diffusive transport occurs in widely different systems ranging from dielectrics and semiconductors through polymers to fractals, glasses, porous and random media. Super-diffusion is comparatively rare and has been observed in porous glasses, polymer chain, biological systems, transport of organic molecules and atomic clusters on surface. One might expect the anomalous heat conduction in media where the anomalous diffusion is observed.

Youssef [15] derived a new formula of heat conduction as follows:

$$q_i + \tau_0 \frac{\partial q_i}{\partial t} = -K I^{\xi-1} \nabla \theta, \quad 0 < \xi b \, 2, \tag{2}$$

Ezzat established a new model of fractional heat conduction equation by using the new Taylor series expansion of time-fractional order, developed by Jumarie [16] as:

$$q_i + \frac{\tau_0^{\xi}}{\xi!} \frac{\partial^{\xi} q_i}{\partial t^{\xi}} = -K \nabla \theta, \quad 0 < \xi \; b \, 1, \tag{3}$$

El-Karamany and Ezzat [17] introduced two general models of fractional heat conduction law for a non-homogeneous anisotropic elastic solid. Uniqueness and reciprocal theorems are proved, and the convolutional variational principle is established and used to prove a uniqueness theorem with no restriction on the elasticity or thermal conductivity tensors except symmetry conditions. The two-temperature dynamic coupled Lord-Shulman and fractional coupled thermoelasticity theories result as limit cases. For fractional thermoelasticity not involving two-temperatures, El-Karamany and Ezzat [18] established the uniqueness, reciprocal theorems and convolution variational principle. The dynamic coupled and Green-Naghdi thermoelasticity theories result as limit cases. The reciprocity relation in case of quiescent initial state is found to be independent of the order of differintegration [17] and [18]. Fractional order theory of a perfect conducting thermoelastic medium not involving two temperatures was investigated by Ezzat and El-Karamany [19]. The finite thermal wave propagation in an infinite half-space under this theory have been studied by Sur and Kanoria [20].

In view of experimental evidence in support of the finiteness of the speed of propagation of heat wave, generalized thermoelasticity theories are more acceptable than the conventional thermoelasticity theories in dealing with practical problems involving very short time intervals and high heat fluxes, like those occurring in laser units, energy channels and nuclear reactor, etc.

Green and Naghdi [21] developed three models for generalized thermoelasticity of homogeneous and isotropic materials which are labeled as Models I, II and III. The nature of those theories are such that when the respective theories are linearized, Model I reduced to the classical heat conduction theory (based on Fourier's law). The linearized versions of Model II and III permit propagation of thermal waves at finite speed. Model II, in particular, exhibits a feature that is not present in the other established thermoelastic models as it does not sustain dissipation of thermal energy [22, 23].

Later, Roychaudhuri [24] established a model of a coupled thermoelasticity theory that includes three phase lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient. The more general model established reduces to the previous models in special cases. According to this model $\dot{q}(P,t+\tau_q) = -K\nabla\dot{\theta}(P,t+\tau_\theta) - K^*\nabla\theta(P,t+\tau_v)$, where K is the thermal conductivity and K^* is the additional material constant. For practical, relevant problems, particularly heat transfer problems involving very short time intervals and the problems of very high heat fluxes, the hyperbolic equation gives significantly different results to the parabolic equation. According to this phenomenon the lagging behavior during heat conduction in solids, should not be ignored, particularly when the elapsed times during a transient process are very small, say about 10^{-7} s or the heat flux is very high. Three-phase-lag (3P) model is very useful for the problems of nuclear boiling, exothermic catalytic reactions, phonon-electron interactions, phonon scattering etc., where the delay time τ_q captures the thermal wave in behavior (a small scale response in time), the phase lag τ_θ captures the effect of phonon-electron interactions (a microscopic response in space), the other delay time τ_v is effective, since in 3P model, the thermal displacement gradient is considered as a constitutive variable. Quintanilla [25] and Racke studied the stability of solutions in three-phase-lag heat conduction, and Quintanilla [26] studied the spatial behavior of solutions for the three-phase-lag heat conduction equation by a semi-infinite cylinder. Different thermoelastic problems have been studied by employing this model of generalized thermoelasticity by several researchers [27–29].

Thermal shocks and very high temperatures inevitably give rise to severe thermal stresses causing catastrophic failure of structural components such as aircraft engines, turbines, space vehicles etc. To avoid such type of failures, functionally graded materials (FGM) are used as discussed by Aboudi at al. [30] and Wetherhold and Wang [31]. These materials are characterized by a microstructure that is spatially variable on a macro scale and were developed initially for high temperature applications. In these materials, the spatial variation of thermal and mechanical properties influences strongly the response to loading. Sugano [32] has presented analytical solution for one dimensional transient thermal stress problem of non-homogeneous plate where the thermal conductivity and Young's modulus vary exponentially, whereas Poisson's ratio and the coefficient of linear thermal expansion vary arbitrarily in the thickness direction. Qian and Batra [33] have studied the problem of a transient thermoelastic deformation of a thick functionally graded plate with edges held at a uniform temperature.

Ghosh and Kanoria [34] studied the thermoelastic response in a functionally graded spherically isotropic infinite elastic medium having a spherical cavity. Kar and Kanoria [35] studied thermoelastic stresses, displacements and temperature distribution in a functionally graded orthotropic hollow sphere due to sudden temperature change on the stress-free boundaries of the hollow sphere in the context of GN-II, GN-III and 3P models of generalized thermoelasticity. Barik et al. [36] have studied a contact problem in FGM. In addition to these reports, thermoelastic analysis in FGM's has been studied by a number of different researchers.

The objective of the contribution is to consider one dimensional thermoelastic disturbance in an infinite isotropic functionally graded medium in the context of three-phase-lag thermoelastic model, GN model II (TEWOED) and GN model III (TEWED), in presence of distributed periodically varying heat sources where the heat equation consists of some nonlocal fractional parameters. All the thermophysical properties of the FGM under consideration are assumed to vary jointly as the exponential power of the space coordinate and a linear function of the temperature. The governing equations for this problem are taken into Laplace-Fourier transform domain. The solutions for thermal displacement, temperature, thermal stress in Laplace transform domain are obtained by taking Fourier inversion which is carried out by using residual calculus, where the poles of the integrand are obtained numerically in complex domain by using Laguerre's method. Then the inversion of Laplace transform has been carried out numerically by applying a method of numerical inversion of Laplace transform based on Fourier series expansion technique [37]. Numerical results for thermal displacement, temperature, thermal stress in physical space-time domain have been obtained for a copper like material and have been presented graphically to show the effect of the fractional order parameter and nonhomogeneity. The dependencies of thermophysical properties on temperature have also been studied.

2 BASIC FORMULATIONS

The stress-strain-temperature relation is

$$\tau_{ij} = 2\mu e_{ij} + \left[\lambda\Delta - \gamma\left(\theta - \theta_0\right)\right]\delta_{ij}, \qquad i, j = 1, 2, 3 \tag{4}$$

where τ_{ij} is the stress tensor, λ, μ are Lame's constants, $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is the coefficient of linear thermal expansion, θ_0 is the reference temperature, θ is the temperature field, the cubical dilatation $\Delta = e_{ii}$ and e_{ij} is the strain tensor given by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{5}$$

Stress equation of motion in absence of body force is

$$\rho \ddot{u}_i = \tau_{ij,j};\ i, j = 1, 2, 3; \tag{6}$$

where u_i $(i = 1, 2, 3)$ are the displacement component and ρ is the density.

Heat equation corresponding to generalized thermoelasticity based on the fractional order three-phase-lag thermoelasticity model [39] is

$$\left[K^* I^{\xi-1}\theta_{,i}\right]_{,i} + \tau_T\left[K\, I^{\xi-1}\ddot{\theta}_{,i}\right]_{,i} + \left[\tau_v^* I^{\xi-1}\dot{\theta}_{,i}\right]_{,i} = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_v\ddot{\theta} + \gamma\theta_0 div\,\ddot{\vec{u}} - \rho\dot{Q}\right) \tag{7}$$

where K^* is an additional material constant, K is the thermal conductivity, c_v is the specific heat at constant strain, Q is the rate of internal heat generation per unit mass, $\tau_v^* = K + K^*\tau_v$; the delay time τ_v is called the phase-lag of thermal displacement gradient. Other delay time τ_T is called the phase-lag of temperature gradient and τ_q is called the phase-lag of the heat flux. Here, dot denotes derivative with respect to time.

For $\tau_q = \tau_T = \tau_v = 0$, Eq. (7) reduces to the GN theory type III and for $\tau_q = \tau_T = \tau_v = 0$ and $K << K^*$ Eq.(7) reduces to GN theory type II.

For functionally graded solid, the parameters λ, μ, K, K^*, γ and τ_v^* are no longer constant but become space and temperature dependent whereas ρ is taken to be space dependent. Thus, we replace λ, μ, K, K^*, γ, ρ and τ_v^* by $\lambda_0 f_1(\vec{x}) f_2(\theta)$, $\mu_0 f_1(\vec{x}) f_2(\theta)$, $K_0 f_1(\vec{x}) f_2(\theta)$, $K_0^* f_1(\vec{x}) f_2(\theta)$, $\gamma_0 f_1(\vec{x}) f_2(\theta)$, $\rho_0 f_1(\vec{x}) f_2(\theta)$ and $\left(\tau_v^*\right)_0 f_1(\vec{x}) f_2(\theta)$ respectively, where λ_0, μ_0, K_0, K_0^*, γ_0 and ρ_0 are assumed to be constants, $f(\vec{x})$ is a given non-dimensional function of the space variable $\vec{x} = (x, y, z)$ and $\left(\tau_v^*\right)_0 = K_0 + K_0^*\tau_v$. Then, the corresponding Eqs. (4), (6) and (7) take the following form

$$\tau_{ij} = f_1(\vec{x}) f_2(\theta)\left[2\mu_0 e_{ij} + \left\{\lambda_0\Delta - \gamma_0\left(\theta - \theta_0\right)\right\}\delta_{ij}\right], \tag{8}$$

$$f_1(\vec{x})\rho_0\ddot{u}_i = f_1(\vec{x}) f_2(\theta)\left[2\mu_0 e_{ij} + \left\{\lambda_0\Delta - \gamma_0\left(\theta - \theta_0\right)\right\}\delta_{ij}\right]_{,j} + f_1(\vec{x})_{,j} f_2(\theta)\left[2\mu_0 e_{ij} + \left\{\lambda_0\Delta - \gamma_0\left(\theta - \theta_0\right)\right\}\delta_{ij}\right], \tag{9}$$

and

$$\begin{aligned}\left(1 + \tau_q\frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho_0 c_v f_1(\vec{x})\ddot{\theta} + \gamma_0\theta_0 f_1(\vec{x}) f_2(\theta)\,\mathrm{div}\,\ddot{\vec{u}} - \rho_0 f_1(\vec{x})\dot{Q}\right) = \left[K_0^* f_1(\vec{x}) f_2(\theta) I^{\xi-1}\theta_{,i}\right]_{,i} \\ + \tau_T\left[K_0 f_1(\vec{x}) f_2(\theta) I^{\xi-1}\ddot{\theta}_{,i}\right]_{,i} + \left[\left(K_0 f_1(\vec{x}) f_2(\theta) + K_0^* f_1(\vec{x}) f_2(\theta)\tau_v\right) I^{\xi-1}\dot{\theta}_{,i}\right]_{,i}\end{aligned} \tag{10}$$

3 FORMULATION OF THE PROBLEM

We now consider a functionally graded infinite isotropic thermoelastic body at a uniform reference temperature θ_0 in the presence of periodically varying heat sources distributed over a plane area. We shall consider one-dimensional disturbance of the medium, so that the thermal displacement vector $\vec{u}$ and the temperature field θ can be expressed in the following form

$$\vec{u} = \left(u(x,t),0,0\right), \qquad \theta = \theta(x,t). \tag{11}$$

It is assumed that material properties depend only on the $x-$coordinate. So, we can take $f_1(\vec{x})$ as $f_1(x)$.

In the context of linear theory of generalized thermoelasticity in absence of body forces based on three-phase lag thermoelasticity model [39] the constitutive equation, strain component, the equation of motion and the heat equation can be written as follows:

$$\tau_{xx} = f_1(x)\, f_2(\theta)\left[\left(\lambda_0+2\mu_0\right)e_{xx}-\gamma_0\left(\theta-\theta_0\right)\right], \tag{12}$$

$$e_{xx}=\frac{\partial u}{\partial x}, \tag{13}$$

$$f_1(x)\, f_2(\theta)\left[\left(\lambda_0+2\mu_0\right)\frac{\partial^2 u}{\partial x^2}-\gamma_0\frac{\partial\theta}{\partial x}\right]+\left[\left(\lambda_0+2\mu_0\right)\frac{\partial u}{\partial x}-\gamma_0\left(\theta-\theta_0\right)\right]\frac{\partial f_1(x)}{\partial x}\, f_2(\theta)=\rho_0\, f_1(x)\frac{\partial^2 u}{\partial t^2}, \tag{14}$$

$$\begin{aligned}&\frac{\partial}{\partial x}\left[K_0^*\, f_1(\vec{x})\, f_2(\theta) I^{\xi-1}\frac{\partial\theta}{\partial x}\right]+\frac{\partial}{\partial x}\left[K_0\tau_T\, f_1(\vec{x})\, f_2(\theta) I^{\xi-1}\frac{\partial\ddot{\theta}}{\partial x}\right]+\frac{\partial}{\partial x}\left[\left(K_0+K_0^*\tau_v\right) f_1(\vec{x})\, f_2(\theta) I^{\xi-1}\frac{\partial\dot{\theta}}{\partial x}\right]\\&\qquad=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho_0 c_v\, f_1(\vec{x})\ddot{\theta}+\gamma_0\theta_0\, f_1(\vec{x})\, f_2(\theta)\,\mathrm{div}\,\ddot{\vec{u}}-\rho_0\, f_1(\vec{x})\dot{Q}\right)\end{aligned} \tag{15}$$

We now introduce the following non-dimensional variables

$$\begin{aligned}&x'=\frac{x}{l},\ \ t'=\frac{vt}{l},\ \ \theta'=\frac{\theta-\theta_0}{\theta_0},\ \ u'=\frac{(\lambda_0+2\mu_0)u}{l\gamma_0\theta_0},\\&\tau'_{x'x'}=\frac{\tau_{xx}}{\gamma_0\theta_0},\ \ e'_{x'x'}=e_{xx},\ \ \tau'_q=\frac{\tau_q v}{l},\ \ \tau'_T=\frac{\tau_T v}{l},\\&\tau'_v=\frac{\tau_v v}{l},\ \ f'_1(x')=f_1(x),\ \ f'_2(\theta)=f_2(\theta),\end{aligned} \tag{16}$$

where l is a standard length and v is a standard speed. Then, after removing primes, Eqs. (12)-(15) can be written in non-dimensional form as follows:

$$\tau_{xx}(x,t)=f_1(x)\, f_2(\theta)\left[\frac{\partial u}{\partial x}-\theta\right], \tag{17}$$

$$e_{xx}(x,t)=\frac{\gamma_0\theta_0}{\lambda_0+2\mu_0}\frac{\partial u}{\partial x}, \tag{18}$$

$$f_1(x)\, f_2(\theta)\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}\right)-\left(\frac{\partial u}{\partial x}-\theta\right)\frac{\partial f_1(x)}{\partial x}\, f_2(\theta)=\frac{f_1(x)}{C_P^2}\frac{\partial^2 u}{\partial t^2}, \tag{19}$$

$$C_T^2\frac{\partial}{\partial x}\left[f_1(x)f_2(\theta)I^{\xi-1}\frac{\partial\theta}{\partial x}\right]+\tau_T C_K^2\frac{\partial}{\partial x}\left[f_1(x)f_2(\theta)I^{\xi-1}\frac{\partial\ddot{\theta}}{\partial x}\right]+(C_K^2+\tau_\nu C_T^2)\frac{\partial}{\partial x}\left[f_1(x)f_2(\theta)I^{\xi-1}\frac{\partial\dot{\theta}}{\partial x}\right]$$
$$=f_1(x)\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left[\frac{\partial^2\theta}{\partial t^2}+\varepsilon_T f_2(\theta)\frac{\partial^3 u}{\partial t^2\partial x}-Q_0\right], \tag{20}$$

where $C_T^2=\dfrac{K_0^*}{\rho_0 c_\nu v^2}$, $C_K^2=\dfrac{K_0}{\rho_0 c_\nu l v}$, $C_P^2=\dfrac{\lambda_0+2\mu_0}{\rho_0 v^2}$, $\varepsilon_T=\dfrac{\gamma_0^2\theta_0}{(\lambda_0+2\mu_0)\rho_0 c_\nu}$, $Q_0=\dfrac{l}{\theta_0 c_\nu v}\dot{Q}$, and it is to be noted that GN theory type III and GN theory type II can be recovered from Eq.(20) by taking $\tau_q=\tau_T=\tau_\nu=0$; and $\tau_q=\tau_T=\tau_\nu=0$, $C_K<<C_T$ respectively.

We assumed that the medium is initially at rest. The undisturbed state is maintained at reference temperature. Then we have

$$u(x,0)=\frac{\partial u(x,0)}{\partial t}=\frac{\partial^2 u(x,0)}{\partial t^2}=\frac{\partial^3 u(x,0)}{\partial t^3}=0,$$
$$\theta(x,0)=\frac{\partial\theta(x,0)}{\partial t}=\frac{\partial^2\theta(x,0)}{\partial t^2}=\frac{\partial^3\theta(x,0)}{\partial t^3}=0. \tag{21}$$

3.1 Exponential variation of nonhomogeneity

We take $f_1(x)=e^{-kx}$ and $f_2(\theta)=1-e^*\theta$, where k is a dimensionless constant and e^* is the empirical material constant. For linearity of the governing partial differential equations of the problem, we have to take into account the condition that $\dfrac{|\theta-\theta_0|}{\theta_0}<<1$, which gives us the approximation of the function $f_2(\theta)$ in the following form $f_2(\theta)\approx f_2(\theta_0)=1-e^*\theta_0$ .Then, the corresponding equations reduce to

$$\tau_{xx}(x,t)=e^{-kx}\left(1-e^*\theta_0\right)\left[\frac{\partial u}{\partial x}-\theta\right], \tag{22}$$

$$e_{xx}(x,t)=\beta_1\frac{\partial u}{\partial x},\qquad \beta_1=\frac{\gamma_0\theta_0}{\lambda_0+2\mu_0}. \tag{23}$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}-k\left(\frac{\partial u}{\partial x}-\theta\right)=\frac{1}{C_P^2}\frac{1}{1-e^*\theta_0}\frac{\partial^2 u}{\partial t^2}, \tag{24}$$

$$C_T^2 I^{\xi-1}\left(\frac{\partial^2\theta}{\partial x^2}-k\frac{\partial\theta}{\partial x}\right)+\tau_T C_K^2 I^{\xi-1}\left(\frac{\partial^2\ddot{\theta}}{\partial x^2}-k\frac{\partial\ddot{\theta}}{\partial x}\right)+(C_K^2+\tau_\nu C_T^2)I^{\xi-1}\left(\frac{\partial^2\dot{\theta}}{\partial x^2}-k\frac{\partial\dot{\theta}}{\partial x}\right)$$
$$=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\frac{1}{1-e^*\theta_0}\left[\frac{\partial^2\theta}{\partial t^2}+\varepsilon_T\left(1-e^*\theta_0\right)\frac{\partial^3 u}{\partial t^2\partial x}-Q_0\right], \tag{25}$$

Let us define Laplace-Fourier double transform of a function $g(x,t)$ by

$$\bar{g}(x,p)=\int_0^\infty g(x,t)e^{-pt}dt,\quad \mathrm{Re}(p)>0,\qquad \hat{\bar{g}}(\alpha,p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{g}(x,p)e^{i\alpha x}dx. \tag{26}$$

Applying Laplace-Fourier double integral transform to the Eqs. (22)-(25) and using the relation (21), we get

$$\hat{\bar{\tau}}_{xx}(\alpha, p) = \left(1 - e^{*}\theta_0\right)\left[-i(\alpha + ik)\hat{\bar{u}}(\alpha + ik, p) - \hat{\bar{\theta}}(\alpha + ik, p)\right], \tag{27}$$

$$\hat{\bar{e}}_{xx}(\alpha, p) = -i\beta_1\hat{\bar{u}}(\alpha, p), \tag{28}$$

$$\left(\alpha^2 + \frac{p^2}{C_P^2\left(1 - e^{*}\theta_0\right)} - ik\alpha\right)\hat{\bar{u}}(\alpha, p) = (i\alpha + k)\hat{\bar{\theta}}(\alpha, p), \tag{29}$$

$$\begin{aligned}&\left[\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)p^{\xi+1} + \left\{\left(C_K^2 + \tau_\nu C_T^2\right)p + \tau_T C_K^2 p^2 + C_T^2\right\}\left(1 - e^{*}\theta_0\right)\alpha^2 - i\alpha k\left\{\left(C_T^2 + p^2 C_K^2\tau_T\right)\right.\right. \\ &\left.\left. + \left(C_K^2 + \tau_\nu C_T^2\right)p\right\}\left(1 - e^{*}\theta_0\right)\right]\hat{\bar{\theta}}(\alpha, p) - i\varepsilon_T p^{\xi+1}\left(1 - e^{*}\theta_0\right)\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)\hat{\bar{u}}(\alpha, p) = \left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)p^{\xi-1}\hat{\bar{Q}}_0\end{aligned} \tag{30}$$

Solving Eqs. (29) and (30) for $\hat{\bar{u}}(\alpha, p)$ and $\hat{\bar{\theta}}(\alpha, p)$ we get

$$\hat{\bar{u}}(\alpha, p) = \frac{\hat{\bar{Q}}_0(i\alpha + k)p^{\xi-1}(1 + p\tau_q + p^2\tau_q^2/2)}{M(\alpha)}, \tag{31}$$

$$\hat{\bar{\theta}}(\alpha, p) = \frac{\hat{\bar{Q}}_0\left(\alpha^2 + \dfrac{p^2}{C_P^2} - i\alpha k\right)(1 + p\tau_q + p^2\tau_q^2/2)}{M(\alpha)}, \tag{32}$$

where

$$\begin{aligned} M(\alpha) &= M_1(p)\alpha^4 + M_2(p)\alpha^3 + M_3(p)\alpha^2 + M_4(p)\alpha + M_5(p) \\ &= M_1(p)(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)(\alpha - \alpha_4), \end{aligned} \tag{33}$$

where $M_1(p),\ M_2(p),\ M_3(p),\ M_4(p),\ M_5(p)$ are given by

$$M_1(p) = \left(1 - e^{*}\theta_0\right)\left[\left(C_K^2 + \tau_\nu C_T^2\right)p + \tau_T C_K^2 p^2 + C_T^2\right], \tag{34}$$

$$M_2(p) = -\left(1 - e^{*}\theta_0\right)\left[2ikp\left(C_K^2 + \tau_\nu C_T^2\right) + 2ik\left(\tau_T C_K^2 p^2 + C_T^2\right)\right], \tag{35}$$

$$\begin{aligned} M_3(p) = {} & p^{\xi+1}(1 + \varepsilon_T)\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)\left(\frac{p^3}{C_P^2} - k^2 p\left(1 - e^{*}\theta_0\right)\right)\left(C_K^2 + \tau_\nu C_T^2\right) - \\ & k^2\left(1 - e^{*}\theta_0\right)\left(\tau_T C_K^2 p^2 + C_T^2\right) + \frac{p^4\tau_T C_K^2}{C_P^2} + \frac{C_T^2}{C_P^2}, \end{aligned} \tag{36}$$

$$M_4(p) = -\left[ikp^{\xi+1}\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)\left\{\varepsilon_T\left(1 - e^{*}\theta_0\right) + 1\right\} + \frac{ikp^2}{C_P^2}\left\{p\left(C_K^2 + \tau_\nu C_T^2\right) + \left(\tau_T C_K^2 p^2 + C_T^2\right)\right\}\right], \tag{37}$$

$$M_5(p) = \frac{p^{\xi+3}}{C_P^2\left(1 - e^{*}\theta_0\right)}\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right). \tag{38}$$

The solutions for thermal stress and strain in Laplace-Fourier transform domain can be obtained from Eqs. (27) and (28) using (31) and (32) as follows:

$$\hat{\bar{\tau}}_{xx}(\alpha, p) = \frac{-\dfrac{p^{\xi+1}}{C_P^2}\left(1 + p\tau_q + \dfrac{p^2\tau_q^2}{2}\right)\hat{\bar{\mathrm{Q}}}_0}{M(\alpha + ik)}, \tag{39}$$

$$\hat{\bar{e}}_{xx}(\alpha, p) = \frac{\beta_1\hat{\bar{\mathrm{Q}}}_0\alpha(\alpha - ik)p^{\xi-1}\left(1 + p\tau_q + \dfrac{p^2\tau_q^2}{2}\right)}{M(\alpha)}, \tag{40}$$

Inverse Fourier transforms of the Eqs. (31), (32), (39) and (40) give the following solutions for thermal displacement, temperature, and thermal stress and strain in Laplace transform domain

$$\bar{u}(x, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{(k + i\alpha)\hat{\bar{\mathrm{Q}}}_0 p^{\xi-1}\left(1 + p\tau_q + p^2\tau_q^2/2\right)e^{-i\alpha x}d\alpha}{M(\alpha)}, \tag{41}$$

$$\bar{\theta}(x, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{\bar{\mathrm{Q}}}_0 p^{\xi-1}\left(1 + p\tau_q + p^2\tau_q^2/2\right)\left\{\alpha^2 + \dfrac{p^2}{C_P^2\left(1 - e^*\theta_0\right)} - i\alpha k\right\}e^{-i\alpha x}d\alpha}{M(\alpha)}, \tag{42}$$

$$\bar{\tau}_{xx}(x, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-\hat{\bar{\mathrm{Q}}}_0\dfrac{p^{\xi+1}}{C_P^2}\left(1 + p\tau_q + p^2\tau_q^2/2\right)e^{-i\alpha x}d\alpha}{M(\alpha + ik)}, \tag{43}$$

$$\bar{e}_{xx}(x, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\beta_1\hat{\bar{\mathrm{Q}}}_0\alpha(\alpha - ik)p^{\xi-1}\left(1 + p\tau_q + p^2\tau_q^2/2\right)e^{-i\alpha x}d\alpha}{M(\alpha)}, \tag{44}$$

where

$$\begin{aligned} M\left(\alpha + ik\right) &= M(-\alpha) \\ &= M_1(p)\alpha^4 - M_2(p)\alpha^3 + M_3(p)\alpha^2 - M_4(p)\alpha + M_5(p) \\ &= M_1(p)\left(\alpha - l_1\right)\left(\alpha - l_2\right)\left(\alpha - l_3\right)\left(\alpha - l_4\right). \end{aligned}$$

3.2 Periodically varying heat source

We assume that the heat source is distributed over the plane $x = 0$ in the following form

$$\begin{aligned} \mathrm{Q}_0 &= \mathrm{Q}_0^*\delta(x)\sin\left(\frac{\pi \mathrm{t}}{\tau}\right) && \text{for } 0\ b\ \mathrm{t}\ b\ \tau \\ &= 0 && \text{for } \mathrm{t} > \tau, \end{aligned} \tag{45}$$

then

$$\hat{\bar{\mathrm{Q}}}_0 = \frac{\mathrm{Q}_0^*\pi\tau\left(1 + e^{-p\tau}\right)}{\sqrt{2\pi}\left(\pi^2 + p^2\tau^2\right)}.$$

Thus, the expressions for thermal displacement, temperature, thermal stress and strain in Laplace transform domain take the following form

$$\bar{u}(x,p)=\int_{-\infty}^{\infty}\frac{Q_0^*\tau(k+i\alpha)(1+e^{-p\tau})p^{\xi-1}\left(1+p\tau_q+p^2\tau_q^2/2\right)e^{-i\alpha x}d\alpha}{2\left(\pi^2+p^2\tau^2\right)M(\alpha)}, \tag{46}$$

$$\bar{\theta}(x,p)=\int_{-\infty}^{\infty}\frac{Q_0^*\tau(1+e^{-p\tau})\left\{\alpha^2+\dfrac{p^2}{C_P^2\left(1-e^*\theta_0\right)}-i\alpha k\right\}p^{\xi-1}\left(1+p\tau_q+p^2\tau_q^2/2\right)e^{-i\alpha x}d\alpha}{2\left(\pi^2+p^2\tau^2\right)M(\alpha)}, \tag{47}$$

$$\bar{\tau}_{xx}(x,p)=\int_{-\infty}^{\infty}\frac{-Q_0^*\tau(1+e^{-p\tau})\dfrac{p^{\xi+1}}{C_P^2}\left(1+p\tau_q+p^2\tau_q^2/2\right)e^{-i\alpha x}d\alpha}{2\left(\pi^2+p^2\tau^2\right)M(-\alpha)}, \tag{48}$$

$$\bar{e}_{xx}(x,p)=\int_{-\infty}^{\infty}\frac{\beta_1 Q_0^*\tau\alpha\left(\alpha-ik\right)(1+e^{-p\tau})p^{\xi-1}\left(1+p\tau_q+p^2\tau_q^2/2\right)e^{-i\alpha x}d\alpha}{2\left(\pi^2+p^2\tau^2\right)M(\alpha)}. \tag{49}$$

Applying contour integration to the Eqs. (46)-(49) we obtain

$$\begin{aligned}\bar{u}(x,p)&=-iQ_0^*\pi\tau N(p)\sum_{\substack{j=1\\ \mathrm{Im}(\alpha_j)<0}}^{4}A_j\left(i\alpha_j+k\right)e^{-i\alpha_j x} \quad \text{for } x>0\\ &=\;\;iQ_0^*\pi\tau N(p)\sum_{\substack{j=1\\ \mathrm{Im}(\alpha_j)>0}}^{4}A_j\left(i\alpha_j+k\right)e^{-i\alpha_j x} \quad \text{for } x<0,\end{aligned} \tag{50}$$

$$\begin{aligned}\bar{\theta}(x,p)&=-iQ_0^*\pi\tau N(p)\sum_{\substack{j=1\\ \mathrm{Im}(\alpha_j)<0}}^{4}A_j\left\{\alpha_j^2+\frac{p^2}{C_P^2\left(1-e^*\theta_0\right)}-i\alpha_j k\right\}e^{-i\alpha_j x} \quad \text{for } x>0\\ &=\;\;iQ_0^*\pi\tau N(p)\sum_{\substack{j=1\\ \mathrm{Im}(\alpha_j)>0}}^{4}A_j\left\{\alpha_j^2+\frac{p^2}{C_P^2\left(1-e^*\theta_0\right)}-i\alpha_j k\right\}e^{-i\alpha_j x} \quad \text{for } x<0,\end{aligned} \tag{51}$$

$$\begin{aligned}\bar{\tau}_{xx}(x,p)&=\frac{iQ_0^*\pi\tau N(p)p^2}{C_P^2}\sum_{\substack{j=1\\ \mathrm{Im}(l_j)<0}}^{4}B_j l_j\left(l_j+ik\right)e^{-il_j x} \quad \text{for } x>0\\ &=-\frac{iQ_0^*\pi\tau N(p)p^2}{C_P^2}\sum_{\substack{j=1\\ \mathrm{Im}(l_j)>0}}^{4}B_j l_j\left(l_j+ik\right)e^{-il_j x} \quad \text{for } x<0,\end{aligned} \tag{52}$$

$$\begin{aligned}\bar{\tau}_{xx}(x,p)&=\frac{iQ_0^*\pi\tau N(p)p^2}{C_P^2}\sum_{\substack{j=1\\ \mathrm{Im}(l_j)<0}}^{4}B_j l_j\left(l_j+ik\right)e^{-il_j x} \quad \text{for } x>0\\ &=-\frac{iQ_0^*\pi\tau N(p)p^2}{C_P^2}\sum_{\substack{j=1\\ \mathrm{Im}(l_j)>0}}^{4}B_j l_j\left(l_j+ik\right)e^{-il_j x} \quad \text{for } x<0,\end{aligned} \tag{53}$$

$$\bar{e}_{xx}(x,p) = -i\beta_1 Q_0^* \pi\tau N(p) \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)<0}}^{4} A_j \alpha_j \left(\alpha_j - ik\right) e^{-i\alpha_j x} \qquad \text{for} \quad x > 0$$
$$= i\beta_1 Q_0^* \pi\tau N(p) \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)>0}}^{4} A_j \alpha_j \left(\alpha_j - ik\right) e^{-i\alpha_j x} \qquad \text{for} \quad x < 0, \tag{54}$$

where A_j ’s and B_j ’s are given by $A_j = \prod_{\substack{n=1 \\ n \neq j}}^{4} \frac{1}{\left(\alpha_j - \alpha_n\right)}$, $B_j = \prod_{\substack{n=1 \\ n \neq j}}^{4} \frac{1}{\left(l_j - l_n\right)}$, $j = 1,2,3,4$ and

$$N(p) = \frac{\left(1+e^{-p\tau}\right) p^{\xi-1} \left(1 + p\tau_q + p^2\tau_q^2/2\right)}{M_1(p)\left(\pi^2 + p^2\tau^2\right)}. \tag{55}$$

4 NUMERICAL INVERSION OF LAPLACE TRANSFORM

Let $\bar{f}(x,p)$ be the Laplace transform of a function $f(x,t)$. Then, the inversion formula for Laplace transform can be written as:

$$f(x,t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{pt} \bar{f}(x,p)\, dp, \tag{56}$$

where d is an arbitrary small real number greater than the real part of all the singularities of $\bar{f}(x,p)$.

Taking $p = d + iw$, the preceding integral takes the form

$$f(x,t) = \frac{e^{dt}}{2\pi} \int_{-\infty}^{\infty} e^{itw} \bar{f}(x, d+iw)\, dw, \tag{57}$$

Expanding the function $h(x,t) = e^{-dt} f(x,t)$ in a Fourier series in the interval $[0, 2T]$ we obtain the approximate formula [37],

$$f(x,t) = f_\infty(x,t) + E_D, \tag{58}$$

where

$$f(x,t) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k \qquad \text{for} \;\; 0 \le t \le 2T \tag{59}$$

and

$$c_k = \frac{e^{dt}}{T}\left[e^{\frac{ik\pi t}{T}} f\left(x, d + \frac{ik\pi t}{T}\right)\right]. \tag{60}$$

The discretization error E_D can be made arbitrary small by choosing d large enough [37]. Since the infinite series in Eq. (59) can be summed up to a finite number N of terms, the approximate value of $f(x,t)$ becomes

$$f_N(x,t) = \frac{1}{2}c_0 + \sum_{k=1}^{N} c_k \qquad \text{for} \quad 0\ b\ t\ b\ 2T. \tag{61}$$

Using the preceding formula to evaluate $f(x,t)$ we introduce a truncation error E_T that must be added to the discretization error. Next, the $\varepsilon-$algorithm is used to accelerate the convergence [37].

The Korrecktur method uses the following formula to evaluate the function $f(x,t)$

$$f(x,t) = f_\infty(x,t) + e^{-2dT} f_\infty(x, 2T+t) + E'_D. \tag{62}$$

where the discretization error $\left|E'_D\right| << \left|E_D\right|$. Thus, the approximate value of $f(x,t)$ becomes

$$f_{NK}(x,t) = f_N(x,t) + e^{-2dT} f_{N'}(x, 2T+t), \tag{63}$$

where N' is an integer such that $N' < N$.

We shall now describe the $\varepsilon-$algorithm that is used to accelerate the convergence of the series in Eq.(61). Let $N = 2q+1,$ where q is a natural number and let $s_m = \sum_{k=1}^{m} c_k$ be the sequence of partial sum of the series in (61).

We define the $\varepsilon-$sequence by $\varepsilon_{0,m} = 0,\ \varepsilon_{1,m} = s_m$ and $\varepsilon_{r+1,m} = \varepsilon_{r-1,m+1} + \dfrac{1}{\varepsilon_{r,m+1} - \varepsilon_{r,m}};\ \ r = 1,2,3,\ldots$

It can be shown that [37] the sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots, \varepsilon_{N,1}$ converges to $f(x,t) + E_D - \dfrac{c_0}{2}$ faster than the sequence of partial sums s_m, $m = 1,2,3,\ldots$

The actual procedure used to invert the Laplace transform consists of using Eq.(69)together with the $\varepsilon-$algorithm. The values of d and T are chosen so according to the criterion outlined in [37].

5 NUMERICAL RESULTS AND DISCUSSIONS

To get the solution for thermal displacement, temperature, thermal stress in space-time domain we have to apply Laplace inversion formula to the Eqs. (48)-(50), respectively. This has been done numerically using a method based on Fourier series expansion technique. To get the roots of the polynomials $M(\alpha)$ and $M(-\alpha)$ in complex domain we have used Lagurre's method. The numerical code has been prepared using Fortran-77 programming language. For computational purpose, copper like material has been taken into consideration. The values of the material constants are taken as follows [38].

$$\varepsilon_T = 0.0168,\ \ \lambda = 1.387\times10^{11}\,\mathrm{N/m^2}, \mu = 0.448\times10^{11}\,\mathrm{N/m^2},\ \ \alpha_t = 1.67\times10^{-8}\,/\mathrm{K}, \theta_0 = 1\mathrm{K}$$

and the hypothetical values of the phase-lag parameters are taken as $\tau_q = 0.001s,\ \ \tau_T = 0.05s,\ \ \tau_v = 0.05s;$ which agrees the stability condition of Quintanilla and Racke [40], that under three-phase-lag heat conduction, if $K^*\tau_q < \tau_v^* < \dfrac{2K\tau_T}{\tau_q}$, where $\tau_v^* = K + K^*\tau_v$, then the solutions are always exponentially stable. Also, we have taken $Q_0^* = 1,\ \tau = 1,\ \ C_P = 1, C_T = 2, C_K = 0.6,$ so that the faster wave is the thermal wave.

In order to study the effect of the fractional order parameter ξ on temperature, thermal stress and displacement distribution, we now present our results in their graphical representation (Figs. 1-3). Figs. 1-3 show the variation of temperature, thermal stress and displacement for three models (GN II, GN III and 3P) for $t = 0.4,\ k = 0.1$ and $\xi = 0.5, 1.0, 1.5$ respectively when $e^* = 0.0005$.

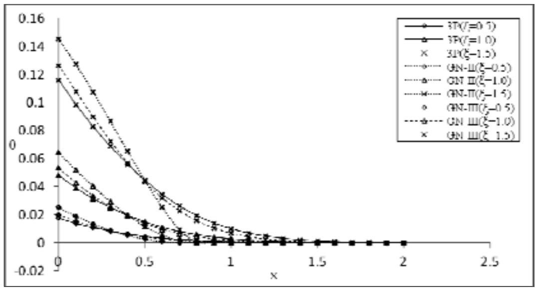

Fig. 1
variation of θ vs. x for $k=1, t=0.4$ and $e^*=0.0005$.

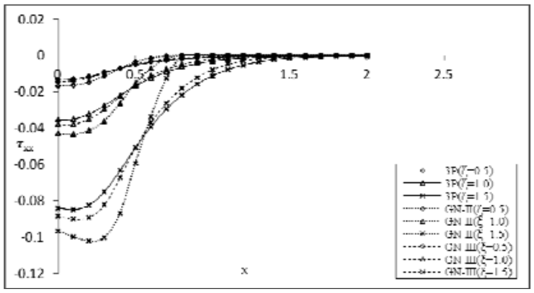

Fig. 2
variation of τ_{xx} vs. x for $k=1, t=0.4$ and $e^*=0.0005$.

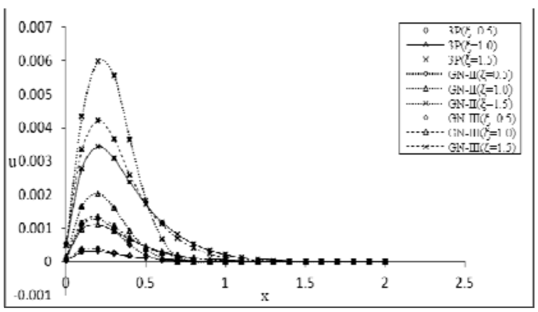

Fig. 3
variation of u vs. x for $k=1, t=0.4$ and $e^*=0.0005$.

Fig.1 depicts the variation of the temperature θ against the distance x for a non-homogeneous material $(k=1)$ and different fractional parameter. It can be seen from the figure that as the value of the fractional order parameter increases, the magnitude of the temperature increases near the plane $x=0$ and ultimately θ approaches to zero. This is because the heat source varies periodically with time for a short duration. This can also be verified from the expression of $\overline{\theta}$ given in Eq. (51) involving $e^{-i\alpha_j x}$, $\mathrm{Im}(\alpha_j)<0$ for $x>0$. For $\xi=1.5$, it is also observed that magnitude of θ for GN II is greater than that of GN III which is again greater than 3P model when $0<x<0.6$. After that, magnitude of θ for GN II falls faster than that of GN III which is again faster than 3P model.

Fig. 2 depicts the variation of the thermal stress τ_{xx} versus the distance x for $t=0.4$ and $k=1$. It can be seen form the figure that τ_{xx} is compressive in nature near the plane $x=0$ where the heat source is active. Also, for $\xi=1.5$, the magnitude of τ_{xx} is greater for GN II model compared to that of GN III, which is again greater than that of 3P model within the range $0<x<0.6$. And for $\xi=1.5$, the magnitude of τ_{xx} decreases sharply for GN II model compared to that of GN III which again decreases sharply compared to that of 3P model.

Fig. 3 shows the variation of thermal displacement u versus the distance x for time $t=0.4$ when $\xi=0.5,1.0,1.5$. This figure shows that the displacement increases to reach its maximum at $x=0.3$ (for GN II), $x=0.25$ (for GN III and 3P) and beyond this, u falls to zero for $\xi=0.5,1.0,1.5$ respectively. It is also seen that in all models, as the fractional parameter ξ increases, the peak of the thermal displacement also increases. The magnitude of the displacement for GN II for a particular range of x $(0<x<0.6)$ is maximum for $\xi=1.5$ than that of $\xi = 1.0$, which is again greater than that for $\xi = 0.5$. The rate of decay in the case of 3P is slower than that of GN III which is again slower than GN II model.

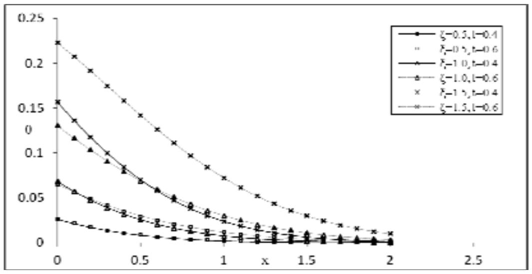

Fig. 4
variation of θ vs. x for $k = 1$, $t = 0.4$, 0.6 and $\xi = 0.5, 1.0, 1.5$.

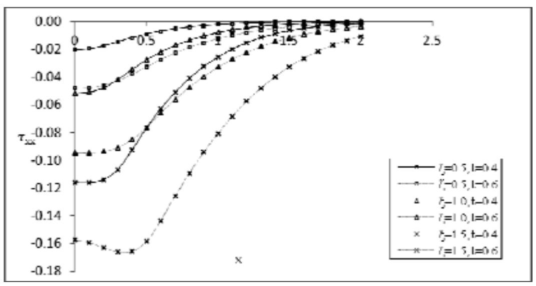

Fig. 5
variation of τ_{xx} vs. x for $k = 1$, $t = 0.4$, 0.6 and $\xi = 0.5, 1.0, 1.5$.

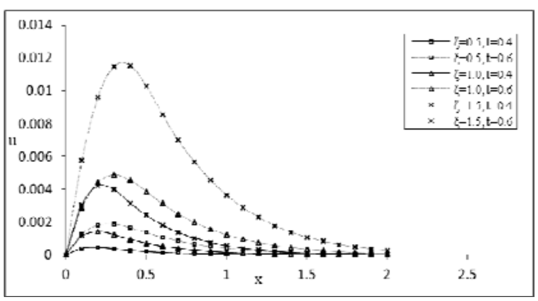

Fig. 6
variation of u vs. x for $k = 1$, $t = 0.4$, 0.6 and $\xi = 0.5, 1.0, 1.5$.

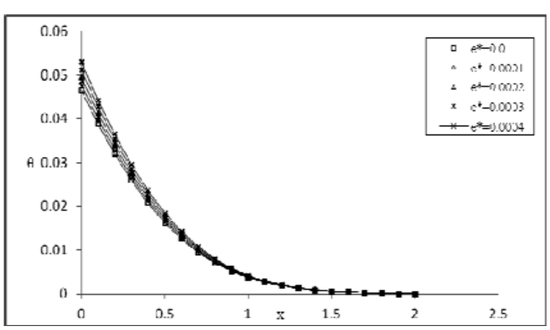

Fig. 7
variation of θ vs. x for $k = 1$, $t = 0.6$ and different e^*.

Figs. 4-6 shows the variation of the temperature, thermal stress and the displacement versus x when $t = 0.4, 0.6$ in case of a non-homogeneous material $(k = 1)$ and $e^* = 0.0005$ for $\xi = 0.5, 1.0$ and 1.5 for GN II model. From the figure it is observed that with the increase of time t, magnitude of θ also increases for different ξ. A similar qualitative behavior is seen in the graphical representations of the thermal stress and the displacement also.

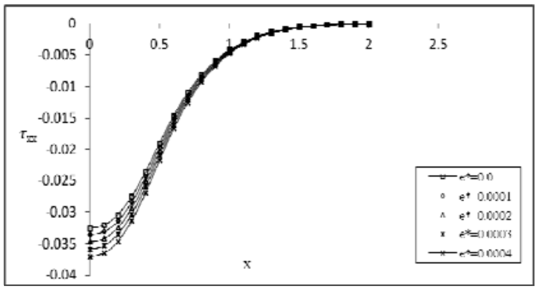

Fig. 8
variation of τ_{xx} vs. x for $k=1$, $t=0.6$ and different e^*.

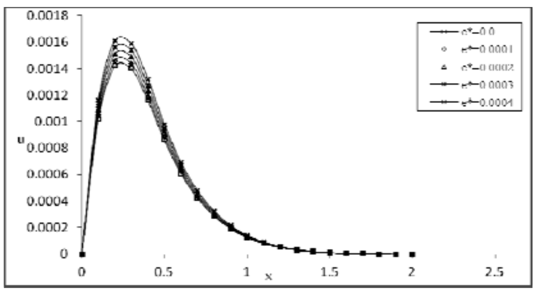

Fig. 9
variation of u vs. x for $k=1$, $t=0.6$ and different e^*.

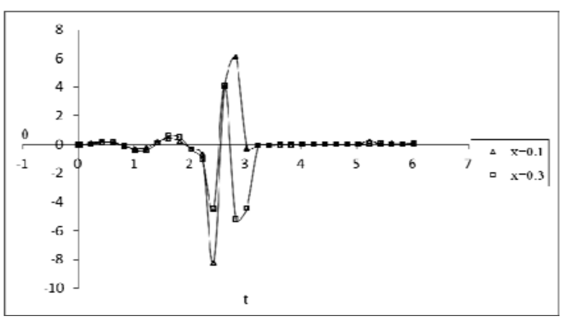

Fig. 10
variation of θ vs. t for $k=6$, $\xi=1.8$ and $x=0.1, 0.3$.

Figs. 7-9 are plotted for GN II model to study the effect of the temperature dependent material parameter e^* on the all thermophysical quantities for weak conductivity $(\xi=0.5)$ and nonhomogeneity parameter $k=1$ and for plotted for $e^*=0.0, 0.0001, 0.0002, 0.0003, 0.0004$. From these figures it seems that rise in magnitude of the material constant will also increase the magnitudes of the profile of the thermophysical quantities. Hence, whenever designing any FGM, dependence of temperature on the elastic parameters have significant effects. Fig.10 depicts the variation temperature (θ) versus t for conductivity parameter $\xi=1.8$ and nonhomogeneity parameter $k=6$ for $x=0.1, 0.3$, respectively. It is evident from the figure that the oscillatory behavior of θ is seen for $0.3<t<3.2$ and after that the temperature almost disappears inside the body. i.e., the thermal wave is propagating with time and with increase of time, it reaches to a steady state.

6 CONCLUSIONS

The present problem of investigating the thermophysical quantities in an isotropic functionally graded material subjected to a periodically varying heat source is studied in the light of generalized fractional order thermoelasticity theories with three relaxation times (3P lag model). The material properties are assumed to vary jointly as exponentially with distance and a linear function of temperature, except the density of the material, which varies only as exponentially with the distance. The analysis of the results permit some concluding remarks.

1. The thermal stress, displacement and temperature have a strong dependency on the non-local fractional order parameter ξ.
2. The dependency of all the elastic constants on temperature has a high significance. So, while designing any FGM, this should be taken into consideration.
3. Here, all the results for $\xi = 1$, $e^* = 0$ and $k = 0$ complies the results of Mallik and Kanoria [41].

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