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Analysis of Mode III Fraction in Functionally Graded Plate with Linearly Varying Properties

M.R. Torshizian*

Mechanical Engineering Department, Mashhad Branch, Islamic Azad University, Mashhad, Iran

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ABSTRACT

A model is provided for crack problem in a functionally graded semi-infinite plate under an anti-plane load. The characteristic of material behavior is assumed to change in a linear manner along the plate length. Also the embedded crack is placed in the direction of the material change. The problem is solved using two separate techniques. Primary, by applying Laplace and Fourier transformation, the governing equation for the crack problem is converted to the solution of a singular integral equation system. Then, finite element technique is employed to analyze this problem by considering quadrilateral eight nodded singular element near the crack tips. The effects of material non-homogeneity and crack length on the stress intensity factor are studied and the results of two methods are judged against each other.

Keywords: Functionally graded material; Stress intensity factor; Linear material properties

1 INTRODUCTION

FUNCTIONALLY Graded Materials (FGM) are designed in a way that it combines strength and thermo-mechanical properties of dissimilar medias to enhance material behaviors in practice especially, in elevated temperature applications. In any FGMs media, existence of non-homogeneity has a pronounced effect on the mechanical behavior of the continuum, particularly when it contains crack. Therefore, in design of any FGM structure the study of fracture behaviors will be an essential part of analysis.

Numerous investigators have studied extensively the problem of a crack behavior in a media made of FGMs. Erdogan et al.[1] have considered the mode I fracture around a crack placed between two bonded isotropic and non-homogeneous half-planes. Erdogan and Ozaturk [2] in their paper reported the mixed boundary value problem for a non-homogeneous medium bonded to a rigid subspace. Sometimes later Wang [3] has analyzed mode III fracture in FG piezoelectric medium containing one crack or multi cracks. Chue and Ou [4] have analyzed the mode III crack in two bonded FG piezoelectric materials. In this research, the crack is assumed to be perpendicular to the interface. Hu et al. [5] have studied the mode III fracture in FG piezoelectric strip located between two different semi-infinite homogeneous plates under mechanical and electric loadings. Ou and Chue [6] in their studies have reported the fracture behavior of an eccentric crack in a strip made of FGP material. Ma et al. [7] have investigated on a FG magneto-electro-elastic strip behavior with an inner crack and lip crack. In another work, Ma and Wu [8] have described a problem for an inner crack normal to the interface of a FG strip bonded to homogeneous substrate structure. Yong and Zhou [9] have studied the behavior of an eccentric crack in FG piezoelectric material layer

* Corresponding author. Tel.: +98 511 6625046; Fax: +98 511 6627560.
E-mail address: torshizian@mshdiau.ac.ir (M.R. Torshizian).

located between two different piezoelectric half-planes. Li et al. [10, 11] have considered a crack in a FGM strip with the material behavior varying in an exponential manner bonded to another substrate with the material behavior varying in a linear manner. Hus and Chue [12] in their reported work have investigated the fracture behavior of angled crack in a functionally graded piezoelectric strip connected to a homogeneous piezoelectric semi-infinite plate. Torshizian and Kargarnovin [13] have analyzed the mode III stress intensity factor at the tips of a crack with arbitrary orientation in a FG strip located between two different half-planes. The effects of orientation and position of the crack and also the variation of material properties on the SIFs have been discussed. Torshizian et al. [14] considered a two-dimensional functionally graded material with an inner crack subjected to an anti-plane external load. The mechanical material properties were assumed to change exponentially in two planar directions and crack orientation was taken to be arbitrary. Kargarnovin et al. [15] studied the fracture mechanics in an FG strip with embedded crack. The material properties were assumed to vary in a linear fashion perpendicular to crack surface under an anti-plane external load.

However, because of involved mathematical complication that could raise due to variation of properties in three directions, almost all of the previous works have considered either an exponential functions for the continuous gradation of the material characteristic properties or have assumed a linear function with gradient of the material properties perpendicular to the crack surface.

In the present study, the anti-plane problem of a crack in a functionally graded semi-infinite plate has been considered. The material characteristic have been considered to change in a linear fashion and crack is considered to be parallel to the direction of which the material properties change. The basic partial differential equations have been inferred and by employing the Laplace and Fourier integral transforms, these equations were reduced into a system of singular integral equations. The values of stress intensity factor (SIF) have been obtained numerically. Then, in a same trend, finite element method has been used to obtain SIFs. Finally, the effects of the length of crack and variation of material properties on the SIFs have been discussed and the obtained results from these two methods have been compared. To the best of author's knowledge for the first time the value of mode III stress intensity factors are computed in a semi-analytical method in an FGM plate with the material properties varying in a linear way where the crack is placed in a direction parallel to the direction of material properties.

2 FORMULATION AND PROBLEM DESCRITION

A functionally graded semi-infinite plate containing an embedded crack with length $2C$ under an external anti-plane shear load is considered (Fig. 1).

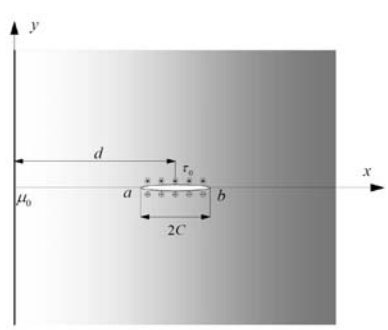

Fig. 1
A crack embedded in a functionally graded semi-infinite plate.

All mechanical material coefficients are assumed to change linearly including shear modulus as following [11]:

$$\mu(x, y) = \mu_0(1+\beta x) \tag{1}$$

where β is a non-homogeneous material parameter in x direction and μ_0 is the shear modulus at $x=0$. The continuity and boundary conditions for the problem respectively:

$$\tau_{xz}(0, y) = 0 \qquad -\infty < y < \infty \tag{2}$$

$$w(x,0)=0 \qquad x<a\ ,\quad x>b \tag{3}$$

$$\tau_{yz}(x,0)=-\tau_0 \qquad a<x<b \tag{4}$$

where τ_0 is a known external anti-plane load effective on the surfaces of crack and $w(x,y)$ is the anti-plane displacement. Moreover, a and b are x-coordinate of the crack tip locations. The constitutive and equilibrium equation in an anti-plane case can be listed, respectively as [3]:

$$\tau_{xz}=\mu(x,y)\frac{\partial w}{\partial x} \tag{5}$$

$$\tau_{yz}=\mu(x,y)\frac{\partial w}{\partial y} \tag{6}$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}=0 \tag{7}$$

By inserting Eqs. (1), (5) and (6) into Eq. (7), the basic partial differential equation will be obtained as:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\beta}{1+\beta x}\frac{\partial w}{\partial x}=0 \tag{8}$$

The problem with representation shown in Fig. 1 can be analyzed by decomposing the domain into an infinite FGM plate with a crack and a FGM semi-infinite plate without crack. Then the governing relations in each case are called. After combining governing relations for two above cases, the main problem can be analyzed. The basic differential equations for each of the above mentioned cases are reviewed in the upcoming sub-sections.

2.1 Analysis of deformation in an infinite FGM plate with crack

Let us begin by introducing the Laplace transform with respect to *y* as:

$$w(x,y)=\int_0^\infty F_1(x,\alpha)\,e^{-\alpha y}\,d\alpha \tag{9}$$

By taking the Laplace transform of Eq. (8), one primarily would get:

$$\int_0^\infty [\frac{d^2F_1}{dx^2}+\frac{\beta}{1+\beta x}\frac{dF_1}{dx}+\alpha^2 F_1]e^{-\alpha y}d\alpha=0 \tag{10}$$

The Eq. (10) will be satisfied if:

$$\frac{d^2F_1}{dx^2}+\frac{\beta}{1+\beta x}\frac{dF_1}{dx}+\alpha^2 F_1=0 \tag{11}$$

Eq. (11) can easily be solved for F_1 as:

$$F_1(x,\alpha)=A(\alpha)\,J_0\left[\alpha(1+\beta x)/\beta\right]+C(\alpha)\;Y_0\left[\alpha(1+\beta x)/\beta\right] \tag{12}$$

where J_0 and Y_0 are the zero order first kind and second kind Bessel functions and also $A(\alpha)$ and $C(\alpha)$ are unknown functions that have to be calculated. Having on hand the F_1 , the anti-plane deformation can be expressed as:

$$w(x,y)=\int_0^\infty \left\{A(\alpha)J_0\left[\alpha(1+\beta x)/\beta\right]+C(\alpha)\,Y_0\left[\alpha(1+\beta x)/\beta\right]\right\}e^{-\alpha y}d\alpha \tag{13}$$

From Eq. (12), for $x=\infty$ anti-plane deformation must be zero, it can be found that $C(\alpha)=0$ and hence, Eq. (13) becomes:

$$w(x,y)=\int_0^\infty A(\alpha)J_0\left[\alpha(1+\beta x)/\beta\right]e^{-\alpha y}d\alpha \tag{14}$$

2.2 Analysis of deformation in a semi-finite FGM plate with no crack

Because of the symmetry, it is sufficient to work only on the upper half of the structure, i.e. $y>0$. Applying Fourier sine transform to Eq. (8), the anti-plane displacement can be calculated as:

$$w(x,y)=\frac{2}{\pi}\int_0^\infty F_2(x,\alpha)\sin(\alpha y)d\alpha \tag{15}$$

Substituting Eq. (15) into Eq. (8) yields:

$$\int_0^\infty\left[\frac{d^2F_2}{dx^2}+\frac{\beta}{1+\beta x}\frac{dF_2}{dx}-\alpha^2F_2\right]\sin(\alpha y)d\alpha=0 \tag{16}$$

Eq. (16) could be satisfied when:

$$\frac{d^2F_2}{dx^2}+\frac{\beta}{1+\beta x}\frac{dF_2}{dx}-\alpha^2F_2=0 \tag{17}$$

Eq. (17) can easily be solved for F_2 as:

$$F_2(x,\alpha)=B(\alpha)K_0\left[\alpha(1+\beta x)/\beta\right]+D(\alpha)I_0\left[\alpha(1+\beta x)/\beta\right] \tag{18}$$

In which I_0 and K_0 are the zero order first kind and second kind modified Bessel functions, and also $B(\alpha)$ and $D(\alpha)$ are some unknown functions ought to be obtained. Having on hand the F_2, the anti-plane deformation can be expressed as:

$$w(x,y)=\frac{2}{\pi}\int_0^\infty\left\{B(\alpha)K_0\left[\alpha(1+\beta x)/\beta\right]+D(\alpha)I_0\left[\alpha(1+\beta x)/\beta\right]\right\}\sin(\alpha y)d\alpha \tag{19}$$

From Eq. (19), for $x=\infty$ anti-plane deformation must be zero, it can be found that $D(\alpha)=0$ and then Eq. (19) can be rewritten as the following:

$$w(x,y)=\frac{2}{\pi}\int_0^\infty B(\alpha)K_0\left[\alpha(1+\beta x)\beta\right]\sin(\alpha y)d\alpha \tag{20}$$

2.3 Analysis of stress and deformation in a cracked FGM semi-infinite plate

The anti-plane deformation analysis of a cracked semi-infinite FGM plate can be obtained as the sum of the anti-plane deformation given by Eq. (14) and Eq. (20):

$$w(x,y)=\int_0^\infty A(\alpha)J_0\left[\alpha(1+\beta x)/\beta\right]e^{-\alpha y}d\alpha+\frac{2}{\pi}\int_0^\infty B(\alpha)K_0\left[\alpha(1+\beta x)/\beta\right]\sin(\alpha y)\,d\alpha \tag{21}$$

After substituting Eq. (21) into Eq. (5) and Eq. (6), the shear stresses τ_{xz} and τ_{yz} can be expressed as:

$$\tau_{xz}(x,y)=-\mu_0(1+\beta x)\left\{\int_0^\infty \alpha A(\alpha)J_1[\alpha(1+\beta x)/\beta]e^{-\alpha y}d\alpha+\frac{2}{\pi}\int_0^\infty \alpha B(\alpha)K_1[\alpha(1+\beta x)/\beta]\sin(\alpha y)d\alpha\right\} \tag{22}$$

$$\tau_{yz}(x,y)=\mu_0(1+\beta x)\left\{\int_0^\infty -\alpha A(\alpha)J_0[\alpha(1+\beta x)/\beta]e^{-\alpha y}d\alpha+\frac{2}{\pi}\int_0^\infty \alpha B(\alpha)K_0[\alpha(1+\beta x)/\beta]\cos(\alpha y)d\alpha\right\} \tag{23}$$

where J_1 and K_1 are the Bessel and modified Bessel functions of first kind and first order, respectively. To determine the unknown function $A(\alpha)$, the dislocation density function, $g(x)$ is defined as [4]:

$$g(x)=\frac{\partial w(x,0)}{\partial x} \tag{24}$$

By introducing $t=(1+\beta x)/\beta$, and recalling Eq. (3) it can be deduced that the function $g(x)$ must fulfill the following single value condition:

$$\int_{a_1}^{b_1} g(t)\,dt=0 \tag{25}$$

In Eq. (25) the parameters a_1 and b_1 are defined as $a_1=a+1/\beta$ and $b_1=b+1/\beta$ where a and b are the crack tip location, respectively (see Fig. 1). By substituting Eq. (21) into Eq. (24), one could be obtained:

$$g(t)=\int_0^\infty \alpha\,A(\alpha)J_1(\alpha t)d\alpha \tag{26}$$

By employing the inverse Hankel transform [16] on Eq. (26) the unknown function $A(\alpha)$ can be obtained as:

$$A(\alpha)=\int_{a_1}^{b_1} t\,J_1(\alpha t)\,g(t)dt \tag{27}$$

To determine the unknown function $B(\alpha)$, the boundary condition given in Eq. (2) can be applied on the Eq. (22). This will yield to:

$$\frac{2}{\pi}\int_0^\infty \alpha\,B(\alpha)K_1(\alpha/\beta)\sin(\alpha y)\,d\alpha=-\int_0^\infty \alpha\,A(\alpha)J_1(\alpha/\beta)\,e^{-\alpha y}d\alpha \tag{28}$$

From Eq. (28) by using inverse Fourier sine transform [17], the unknown function $B(\alpha)$ could be obtained as:

$$B(\alpha)=\frac{-1}{K_1(\alpha/\beta)}\int_0^\infty \frac{s}{s^2+\alpha^2}A(s)J_1(s/\beta)\,ds \tag{29}$$

Substituting Eq. (27) into Eq. (29), $B(\alpha)$ can be obtained as follows:

$$B(\alpha)=\frac{-1}{K_1(\alpha/\beta)}\int_{a_1}^{b_1} f(\alpha,t)\,g(t)dt \tag{30}$$

where the $f(\alpha,t)$ is:

$$f(\alpha,t)=\int_0^{\infty}\frac{st}{s^2+\alpha^2}J_1(s/\beta)J_1(st)\,ds \tag{31}$$

Substituting Eq. (27) and Eq. (30) into Eq. (23) the shear stress τ_{yz} can be expressed as:

$$\begin{aligned}\tau_{yz}(x,y)=\mu_0(1+\beta x)\int_{a_1}^{b_1}\int_0^{\infty}-\alpha\, t\, J_1(\alpha t)\,J_0\left[\alpha(1+\beta x)/\beta\right]e^{-\alpha y}d\alpha.\,g(t)\,dt-\\ \mu_0(1+\beta x)\frac{2}{\pi}\int_{a_1}^{b_1}\int_0^{\infty}\frac{\alpha K_0\left[\alpha(1+\beta x)/\beta\right]}{K_1(\alpha/\beta)}f(\alpha,t)\cos(\alpha y)\,d\alpha\,.g(t)dt\end{aligned} \tag{32}$$

From Eq. (32) with the boundary condition given in Eq. (4), the following integral equation is obtained:

$$\tau_{yz}(x,0)=-\tau_0=-\mu_0(1+\beta x)\left\{\int_{a_1}^{b_1}H_1(x,t)g(t)dt+\frac{1}{\pi}\int_{a_1}^{b_1}H_3(x,t)g(t)dt\right\} \tag{33}$$

The kernels $H_1(x,t)$ and $H_3(x,t)$ are as following:

$$H_1(x,t)=\int_0^{\infty}\alpha\, t\, J_0\left[\alpha(1+\beta x)/\beta\right]J_1(\alpha t)d\alpha \tag{34}$$

$$H_3(x,t)=2\int_0^{\infty}\frac{\alpha\, K_0\left[\alpha(1+\beta x)/\beta\right]}{K_1(\alpha/\beta)}f(\alpha,t)d\alpha \tag{35}$$

The kernel $H_1(x,t)$ is an even function and could be rewritten as the following form:

$$H_1(x,t)=\frac{1}{2}\int_{-\infty}^{\infty}h(\alpha,x,\,t)d\alpha \tag{36}$$

where the function $h(\alpha,x,t)$ is:

$$h(\alpha,x,t)=\alpha\, t\, J_0\left[\alpha(1+\beta x)/\beta\right]J_1(\alpha t) \tag{37}$$

The function $h(\alpha,x,t)$ can be broken into two "singular" and "non-singular" parts as [18]:

$$h(\alpha,x,t)=\underbrace{h_{\infty}(\alpha,x,t)}_{\text{singular}}+\underbrace{[h(\alpha,x,t)-h_{\infty}(\alpha,x,t)]}_{\text{non singular}} \tag{38}$$

In which, $h_{\infty}(\alpha,x,t)$ is the asymptotic value of $h(\alpha,x,t)$ for $\alpha\to\infty$. By looking at the asymptotic solution $h_{\infty}(\alpha,x,t)$ can be expressed as:

$$h_{\infty}(\alpha,x,t)=\frac{1}{\pi}\frac{\alpha}{|\alpha|}\left\{\sin\left[\alpha(t-x-1/\beta)\right]-\cos\left[\alpha(t+x+1/\beta)\right]\right\} \tag{39}$$

By adding and subtracting $h_{\infty}(\alpha,x,t)$ to and from the integrand, the kernel $H_1(x,t)$, would be determined as:

$$H_1(x,t)=\frac{1}{\pi}\left[\frac{1}{t-x-1/\beta}+H_2(x,t)\right] \tag{40}$$

where in which the function $H_2(x,t)$ is defined as:

$$H_2(x,t)=\int_0^{\infty}\left\{\pi\alpha t J_0\left[\alpha(1+\beta x)/\beta\right]J_1(\alpha t)-\sin\left[\alpha(t-x-1/\beta)\right]\right\}d\alpha \tag{41}$$

Substituting Eq. (40) into Eq. (33), the following singular integral equation could be obtained:

$$\tau_0=\mu_0(1+\beta x)\frac{1}{\pi}\int_{a_1}^{b_1}\left[\frac{1}{t-x-1/\beta}+H_2(x,t)+H_3(x,t)\right]g(t)dt \tag{42}$$

3 SINGULAR INTEGRAL EQUATION SOLUTION

It has been reported that Eq. (42) cannot be solved in a closed form way hence; it has to be solved numerically[2]. The interval are normalized by suggesting the linear transforms:

$$x=\frac{b-a}{2}r+\frac{b+a}{2} \qquad a<x<b \quad\rightarrow\quad -1<r<1 \tag{43}$$

$$t=\frac{b_1-a_1}{2}u+\frac{b_1+a_1}{2} \qquad a_1<t<b_1 \quad\rightarrow\quad -1<u<1 \tag{44}$$

$$g(t)=g\left(\frac{b_1-a_1}{2}u+\frac{b_1+a_1}{2}\right)=G(u) \tag{45}$$

$$f(r)=\frac{\tau_0}{\mu_0(1+\beta x)}=\frac{2\tau_0}{\mu_0\beta\left[(b-a)r+(b+a)\right]} \tag{46}$$

$$H_2(r,u)=\frac{b_1-a_1}{2}H_2(x,t) \qquad , \qquad H_3(r,u)=\frac{b_1-a_1}{2}H_3(x,t) \tag{47}$$

Substituting Eqs. (43)-(47) in Eq. (42) and normalizing the result yields to:

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{u-r}+H_2(r,u)+H_3(r,u)\right]G(u)du=f(r) \tag{48}$$

The unknown function $G(u)$ can be converted to the following using Chebyshev polynomials express [2]:

$$G(u)=\frac{1}{\sqrt{1-u^2}}\sum_{n=0}^{\infty}c_nT_n(u) \tag{49}$$

where the coefficients c_n are unknown and $T_n(u)$ is the first kind Chebyshev polynomial. By inserting $G(u)$ into Eq.(48) and using following Chebyshev polynomials properties:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(u)\,du}{(u-r)\sqrt{1-u^2}}=\begin{cases}U_{n-1}(r) & |r|<1\ ,\ (n=1,2,...)\\ -\dfrac{|r|/r}{\sqrt{r^2-1}}[r-(|r|/r)\sqrt{r^2-1}]^n & |r|>1\ ,\ (n=0,1,..)\end{cases} \tag{50}$$

Eq. (48) can be converted to the following algebraic equation:

$$\sum_{n=1}^{\infty}c_n\{U_{n-1}(r)+\frac{1}{\pi}\int_{-1}^{1}\left[H_2(r,u)+H_3(r,u)\right]\frac{T_n(u)}{\sqrt{1-u^2}}du\}=f(r) \tag{51}$$

Also, by employing the orthogonality condition on $T_n(u)$ in conjunction with Eq. (25) it follows that $c_0=0$ [13]. The series of Eq. (51) would be truncated a $n=N$**,** using the following discrete values of the variables u_j [2]:

$$T_N(r_i)=0\quad ,\quad r_i=\cos[\frac{\pi}{2N}(2i-1)]\quad ,\quad i=1,2,...,N \tag{52}$$

To integrate $H_2(r,u)$ and $H_3(r,u)$ on the range of $(0,+\infty)$ the Gauss-Kronrod quadrature method [19] is applied. After substituting $r=r_i$, $i=1,\ldots,N$ in Eq. (51), a set of linear algebraic equation in terms of $c_1,\ldots,c_N$ will be obtained out of which c_i coefficients can be calculated. Substituting Eq. (50) for $|r|>1$ into Eq. (33), the stress component can be obtained as:

$$\tau_{yz}(x,0)=\mu_0(1+\beta x)\sum_{n=1}^{\infty}c_n\{\frac{-|r|/r}{\sqrt{r^2-1}}[r-\frac{|r|}{r}\sqrt{r^2-1}]^n+\frac{1}{\pi}\int_{-1}^{1}\left[H_2(r,u)+H_3(r,u)\right]\frac{T_n(u)}{\sqrt{1-u^2}}du\} \tag{53}$$

The mode III stress intensity factor $K_{\text{III}}(a),K_{\text{III}}(b)$ at the crack tips $x=a$, $x=b$, respectively could be defined as [12]:

$$K_{\text{III}}(a)=\lim_{x\to a}\sqrt{2(a-x)}\,\tau_{yz}(x,0) \tag{54}$$

$$K_{\text{III}}(b)=\lim_{x\to b}\sqrt{2(x-b)}\,\tau_{yz}(x,0) \tag{55}$$

Substituting Eq. (53) into Eqs. (54) and (55), $K_{\text{III}}(a),K_{\text{III}}(b)$ are calculated as:

$$K_{\text{III}}(a)=\mu_0(1+\beta a)\sqrt{\frac{b-a}{2}}\sum_{n=1}^{\infty}(-1)^n c_n \tag{56}$$

$$K_{\text{III}}(b)=-\mu_0(1+\beta b)\sqrt{\frac{b-a}{2}}\sum_{n=1}^{\infty}c_n \tag{57}$$

The normalized stress intensity factors are defined by dividing $K_{\text{III}}(a),K_{\text{III}}(b)$ by $\tau_0\sqrt{(b-a)/2}$ as follow:

$$\bar{K}_{\text{III}}(a)=\left[\mu_0(1+\beta a)/\tau_0)\right]\sum_{n=1}^{\infty}(-1)^n c_n \tag{58}$$

$$\bar{K}_{\mathrm{III}}(b) = -\left[\mu_0(1+\beta b)/\tau_0)\right]\sum_{n=1}^{\infty} c_n \tag{59}$$

4 NUMERICAL RESULTS, VERIFICATION AND DISCUSSION

4.1 Semi-analytical solution

As previously seen, the mathematical model for the fracture problem of a FGM semi-infinite plate with the geometrical representation and loading is shown in Fig. 1. Based on the Fourier and Laplace integral transforms techniques a semi-analytical solution is developed. Then, the governing equation of the crack problem is converted to a system of singular integral equation, which is subsequently replaced by a set of linear algebraic equation in terms of c_i. The coefficients c_i in Eq. (51) are calculated by means of a self-developed computer program using MATLAB solver. Furthermore, stress intensity factors at the crack tips for the third mode are obtained and the effects of non-homogeneous material parameters and crack location on SIFs at crack tips are analyzed.

Figs. 2 and 3 illustrate the variations of the normalized third mode, stress intensity factors with relative crack length d/c for different gradient parameter βc. The results show that the normalized SIFs have higher values at the crack tips with higher stiffness. These curves show that for the equal length cracks, $\bar{K}_{III}(a)$ decreases and $\bar{K}_{III}(b)$ increases as the βc increases. When the crack is closer to free surface, the difference between $\bar{K}_{III}(a)$ and $\bar{K}_{III}(b)$ increases and with increases d/c, the difference between $\bar{K}_{III}(a)$ and $\bar{K}_{III}(b)$ decreases. Because the relative different material property, in the right and left sides of crack tip, becomes smaller with increases d/c.

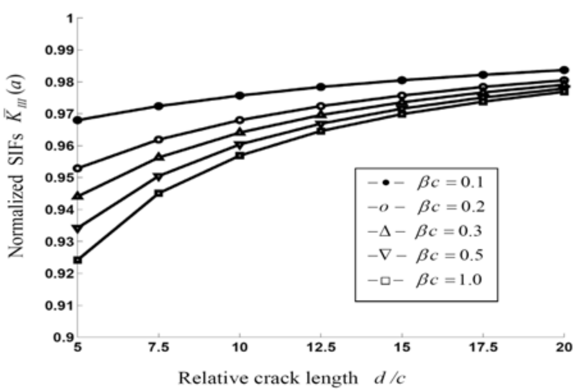

Fig. 2
Variations of normalized SIFs at crack tip a vs. d/c for different βc.

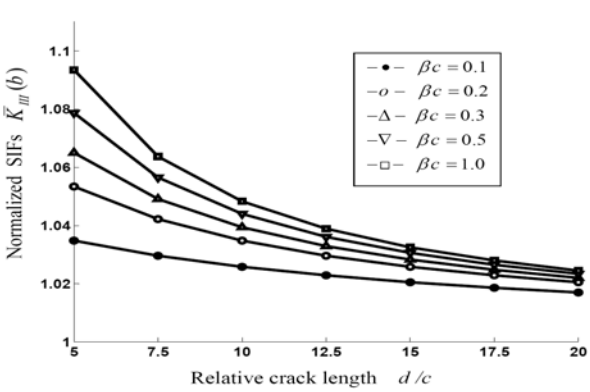

Fig. 3
Variations of normalized SIFs at crack tip b vs. d/c for different βc.

4.2 Finite element modeling

The calculations obtained from the analytical method are verified by comparing the finite element analysis. In finite element modeling, a square plate is considered comprising of 720 quadrilateral 8-noded elements (see Fig. 4). Notice that in order to get better results, near the crack tips very fine mesh is generated. To model a finite crack in a semi-infinite plate, a square plate with side length of $20C$ is considered. Moreover, the elements around the crack are chosen to be the singular types, which simulate suitably the singularity condition at crack tips. It is assumed that an equivalent anti-plane external loading of τ_0 is directly applied around the crack face surfaces. Once the displacement field is determined, the mode III stress intensity factor can be calculated (see Ref. 14).

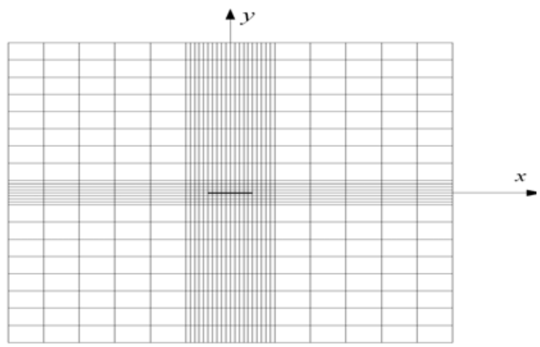

Fig. 4
Finite element model of a FGM square plate with a central crack.

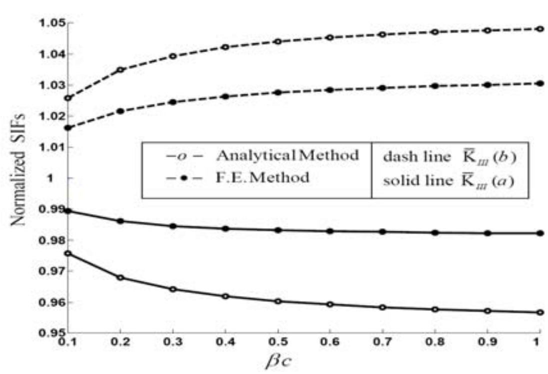

Fig. 5
Variations of normalized SIFs at crack tips vs. βc for at $d/c = 10$.

Fig. 5 shows comparison of the normalized stress intensity factors with different gradient parameter βc by the analytical method solution with finite element method. The maximum difference between the results using two different methods is less than 3% for $d/c = 10$ at $\beta c = 1$. It shows an excellent agreement between the analytical and finite element results.

5 CONCLUSIONS

The anti-plane problem of a crack located in a semi-infinite plate made of functionally graded material is investigated. The properties of material are assumed to vary in a linear fashion. The crack direction is parallel to the direction of the material properties variation. Fourier and Laplace transformers are used to convert the partial differential equations into a singular integral equations system. This integral equation is then solved using Gauss-Chebyshev polynomials. The same problem has been reworked using finite element method in which around each crack tip four singular elements are used. The rest of plate is descretized using 8-noded quadrilateral elements. Several different examples are solved and effects of material non-homogeneity and crack location on the values of the stress intensity factors are discussed. It is observed that, for the equal length crack, stress intensity factors increases if non-homogeneity of the material increases. Because the relative different material property, in the left and right sides of crack tip, becomes greater. When the crack is located closer to the free surface, the difference between the third mode stress intensity factors at the right and left sides of the crack tips become more remarkable.

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