

Effect of Exponentially-Varying Properties on Displacements and Stresses in Pressurized Functionally Graded Thick Spherical Shells with Using Iterative Technique

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ABSTRACT

A semi-analytical iterative method as one of the newest analytical methods is used for the elastic analysis of thick-walled spherical pressure vessels made of functionally graded materials subjected to internal pressure. This method is accurate, fast and has a reasonable order of convergence. It is assumed that material properties except Poisson's ratio are graded through the thickness direction of the sphere according to an exponential distribution. For different values of inhomogeneity constant, distributions of radial displacement, radial stress, circumferential stress, and von Mises equivalent stress, as a function of radial direction, are obtained. A numerical solution, using finite element method (FEM), is also presented. Good agreement was found between the semi-analytical results and those obtained through FEM.

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Keywords: Iterative technique; Elastic analysis; Functionally graded material (FGM); Thick sphere; Exponential.

1 INTRODUCTION

FUNCTIONALLY graded materials (FGMs) are a special group of heterogeneous composite materials with mechanical properties changing continuously from one surface to another. These materials are advanced, resistant to heat, erosion and corrosion, and have high fracture toughness [1]. Advances in material synthesis technologies have spurred the development of FGM with promising applications in aerospace, transportation, energy, cutting tools, electronics, and biomedical engineering [2]. Although the functionally graded materials (FGMs) are extensively employed to sustain elevated temperatures and severe temperature gradients, they have been mainly proposed to precisely monitor the distribution of the material properties within the stressed components to meet the local strength requirements [3]. Closed-form solutions are obtained by Tutuncu and Ozturk [4] for cylindrical and spherical vessels with variable elastic properties obeying a simple power law through the wall thickness which resulted in simple Euler-Cauchy equations whose solutions were readily available. Elastic analysis of internally pressurized thick walled spherical pressure vessels of functionally graded materials was studied by You et al. [5]. Two kinds of pressure vessel are considered: one consists of two homogeneous layers near the inner and outer surfaces of the vessel and one functionally graded layer in the middle; the other consists of the functionally graded material only. Based on the assumption that Poisson's ratio is constant and the modulus of elasticity is an

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exponential function of radius, Chen and Lin [6] analyzed stresses and displacements in FG cylindrical and spherical pressure vessels by a numerical method. A hollow sphere made of FGMs subjected to radial pressure was analyzed by Li et al. [7]. This paper reduces the problem to a Fredholm integral equation. Using plane elasticity theory and complementary functions method, Tutuncu and Temel [8] obtained axisymmetric displacements and stresses in functionally graded hollow cylinders, disks and spheres subjected to uniform internal pressure. Complementary functions method reduces the boundary value problem to an initial-value problem. The boundary value problem can be solved accurately by Runge–Kutta method. Zamani Nejad et al. [9] developed a 3- D set of field equations of FGM thick shells of revolution in curvilinear coordinate system by tensor calculus. Assuming that material properties vary nonlinearly in the radial direction, and Poisson's ratio is constant, exact solutions for stresses and displacement in a functionally graded (FGM) pressurized thick-walled hollow circular cylinder are obtained under generalized plane strain and plane stress assumptions, respectively. Deformations and stresses inside multilayered thick-walled spheres are investigated by Borisov [10]. Each sphere is characterized by its elastic modules. The zone of contact between each of the spheres is continuous on the surface. Assuming the volume fractions of two phases of an FG material (FGM) vary only with the radius, Nie et al. [11] obtained a technique to tailor materials for functionally graded linear elastic hollow cylinders and spheres to attain through-the-thickness either a constant circumferential stress or a constant in-plane shear stress. Assuming that the modulus of elasticity varies nonlinearly in the radial direction as power function of r , and Poisson's ratio is constant, Ghannad and Zamani Nejad [12] derived the governing equations for axisymmetric thick spherical shells made of nonhomogeneous functionally graded materials (FGMs) subjected to internal and external pressure in the general case. The analytical solution of the equations for real, double and complex roots was obtained. Zamani Nejad et al. [13] obtained stresses and the displacements in thick-walled spherical shells made of functionally graded materials subjected to internal and external pressure are developed by an analytical method.

Generally speaking, the analysis of actual engineering problems involves solution of nonlinear differential equations or linear differential equations with variable coefficients. Except for a limited number, these problems cannot be solved explicitly and normally fail to yield exact solutions [14]. Therefore, the researchers are looking for semi-exact and analytical methods. The Adomian's decomposition method (ADM) [15] was presented in 1988 by Adomian. This method had been introduced as a semi-exact and approximation solution for partial differential equations (PDE's) and ordinary differential equations (ODE's). After 1988, many researchers tried to perfect this method. However, He presented the homotopy perturbation method (HPM) [16-21]. The benefits of this method, compared to the previous methods, are that it is more accurate, requires little computational volume, is easier to use, etc. There are many standard methods for solving nonlinear ordinary differential equations, such as Lie Group method [22], Inverse Scattering method [23], Hirota's Bilinear method [24], Homogeneous Balance method [25], and Variational Iteration method (VIM), that was proposed by He [26-29]. The semi-analytical solution has been used by many authors to solve various types of linear and non-linear equations in solid mechanics. Hojjati and Jafari [30] studied the elastic analysis of non-uniform thickness and density of rotating disks subjected to only centrifugal loadings by variational iteration method (VIM). Hojjati and Jafari [31] also studied the elastic analyses of non-uniform thickness and density of rotating disks under only centrifugal body loadings. They used two semi-exact methods, namely the homotopy perturbation and Adomian's decomposition. Still in another study, Hojjati and Jafari [32] presented analytical solutions for the elastic-plastic stress distribution in rotating annular disks with variable thickness and density under the plane stress assumption based on Tresca's yield criterion, its associated flow rule and linear strain hardening by applying the homotopy perturbation method. They carried out their research only for centrifugal loading. Asghari and Ghafoori [33] presented a semi-analytical three-dimensional elasticity solution for rotating functionally graded disks for both hollow and solid disks. Their objective was to generalize an available two-dimensional plane-stress solution to a three-dimensional one in the elastic zone.

Semi-analytical iterative technique is one of the non-perturbation methods that do not require any small/large parameter. An approach in which an attempt is made to incorporate the various tools into a problem solvers disposal, a combination of analytical, symbolic and numerical computation, has been considered by [34, 35].

In this paper, using the semi-analytical solution, an analysis of displacements and stresses of FGM thick spherical pressure vessels with exponential varying material properties is presented. For validation purposes, a numerical solution based on a commercial finite element program ANSYS 12 is used. Compliance of the two types of simulations verifies the current study.

2 PROBLEM FORMULATION AND SOLUTION METHOD

Consider a thick FGM spherical pressure vessel, with internal radius a and external radius b . The sphere is assumed to be under the action of a constant pressure P , on the inner surface (Fig. 1).

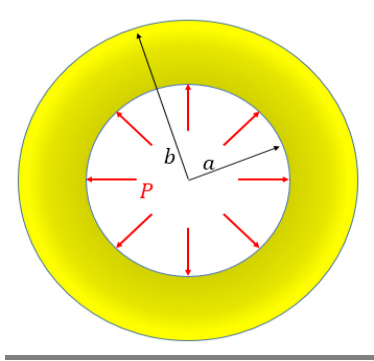


Fig. 1
Cross-section of thick sphere.

Spherical coordinates (r, θ, ϕ) are considered. Infinitesimal deformations are presumed. Besides, in this study, Poisson's ratio " ν " is taken to be a constant value, and also the modulus of elasticity E , varies radially according to exponential form, as follows:

$$E = E_i e^{\left(\frac{r-a}{b-a}\right)n} \quad (1)$$

Here, E_i is the module of elasticity at $r = a$ and n is the inhomogeneity constant.

In the spherical symmetric case ($\sigma_\theta = \sigma_\phi$), the equilibrium equation in the absence of body forces is expressed as:

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \quad (2)$$

where σ_r and σ_θ are radial and circumferential stresses, respectively.

The displacement in the r -direction is denoted by $u(r) = u$. Two strain components can be expressed as:

$$\varepsilon_r = \frac{du}{dr} \quad (3)$$

$$\varepsilon_\theta = \frac{u}{r} \quad (4)$$

where ε_r and ε_θ are radial and circumferential strains, respectively.

The relationships between the strains and stresses can be described by Hooke's law:

$$\begin{bmatrix} \sigma_r \\ \sigma_{\theta=\phi} \end{bmatrix} = \lambda e^{mr} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{bmatrix} \quad (5)$$

where

$$\lambda = \frac{E_i}{(1+\nu)(1-2\nu)} e^{-\left(\frac{a}{b-a}\right)^n} \quad (6)$$

$$m = \frac{n}{b-a} \quad (7)$$

Substitution of Eqs. (3), (4) and (5) into Eq. (2) leads to the Navier equation in terms of displacement as follows:

$$r^2 \frac{d^2 u}{dr^2} + (mr^2 + 2r) \frac{du}{dr} + (\beta r - 2)u = 0 \quad (8)$$

where;

$$\beta = \frac{2n\nu}{(1-\nu)(b-a)} \quad (9)$$

Boundary condition is as follows:

$$\begin{cases} \sigma_r|_{r=a} = -P \\ \sigma_r|_{r=b} = 0 \end{cases} \quad (10)$$

Since Eq. (8) is considered as a highly nonlinear one, and it has not been solved analytically so far, our elaboration in this paper is to solve the so-called problem using a powerful method called iterative simulation to find $u(r)$.

2.1 Basic idea of iterative technique [34]

Any differential equation can be written as:

$$L(u(x)) + N(u(x)) - g(x) = 0 \quad (11)$$

Along with boundary condition

$$B\left(u, \frac{du}{dx}\right) = 0 \quad (12)$$

where x denotes the independent variable, $u(x)$ is an unknown function, $g(x)$ is a known function, L is a linear operator, N is a nonlinear operator and B is a boundary operator. Of course, the main requirement here is that L must be the linear part of the ODE but it is possible to take some linear parts and add them to N , as needed. The method we propose works in the following way: we start by assuming that $u_0(x)$ is an initial guess of the solution to the problem $u(x)$ and is the solution of the equation.

$$L(u_0(x)) = f(x) \quad , \quad B\left(u_0, \frac{du_0}{dx}\right) = 0 \quad (13)$$

To generate the next improvement to the solution, i.e., the next iterate, we solve the following problem:

$$\begin{cases} L(u_1(x)) + N(u_1(x)) = g(x) \\ B\left(u_1, \frac{du_1}{dx}\right) = 0 \end{cases} \quad (14)$$

and thus we have a simple iterative procedure which is effectively the solution of a linear set of problems i.e.,

$$\begin{cases} L(u_{n+1}(x)) + N(u_n(x)) = g(x) \\ B\left(u_{n+1}, \frac{du_{n+1}}{dx}\right) = 0 \end{cases} \quad (15)$$

It is important to note that each of the $u_i(x)$ is standing alone solutions to problem (11). We believe that this iterative procedure, although very simple to apply, has merits in that each solution is an improvement over the previous iterates and as more and more iterations are taken, the solution converges to the solution of problem (11).

2.2 Implementation of iterative technique

By applying the iterative technique to Eq. (8), the differential equation of thick-wall sphere for the radial displacement is separated into the linear and nonlinear parts. Therefore, we will have:

$$L(u) = r^2 \frac{d^2 u}{dr^2} + 2r \frac{du}{dr} - 2u \quad (16)$$

$$N(u) = (mr^2) \frac{du}{dr} + (\beta r)u \quad (17)$$

Thus, the initial problem is

$$L(u_0) = 0 \quad \text{with} \quad \begin{cases} \lambda e^{ma} \left[(1-\nu) \frac{du_0}{dr} \Big|_{r=a} + 2\nu \frac{u_0}{a} \right] = -P \\ \lambda e^{mb} \left[\nu \frac{du_0}{dr} \Big|_{r=b} + \frac{u_0}{b} \right] = 0 \end{cases} \quad (18)$$

The general solution of initial problem Eq. (18) is as follows:

$$u_0 = \frac{C_1}{r^2} + C_2 r \quad (19)$$

Using the boundary conditions, the integration constants C_1 and C_2 can be obtained as:

$$C_1 = \frac{a^3 b^3 e^{-am} P}{2\lambda(b^3 - a^3)(1-2\nu)} \quad (20)$$

$$C_2 = \frac{a^3 e^{-am} P}{\lambda(b^3 - a^3)(1+\nu)} \quad (21)$$

The second iteration can be carried through and is given as:

$$r^2 \frac{d^2 u_1}{dr^2} + 2r \frac{du_1}{dr} - 2u_1 = -(mr^2) \frac{du_0}{dr} - (\beta r)u_0 \quad \text{with} \quad \begin{cases} \lambda e^{\beta_1 a} \left[(1-\nu) \frac{du_1}{dr} \Big|_{r=a} + 2\nu \frac{u_1}{a} = -P \right. \\ \left. \lambda e^{\beta_1 b} \left[\nu \frac{du_1}{dr} \Big|_{r=b} + \frac{u_1}{b} = 0 \right. \right. \end{cases} \quad (22)$$

and has solution:

$$u_1 = C_4 r + \frac{C_3}{r^2} - \frac{C_2}{4} (\beta + m)r^2 + \frac{C_1 (\beta - 2m)}{2r} \quad (23)$$

Using the boundary conditions, the integration constants C_3 and C_4 can be obtained as:

$$C_3 = \frac{a^3 e^{-am}}{4\lambda(b^3 - a^3)(1-2\nu)} \left(\begin{array}{l} e^{am} \lambda (b^4 C_2 (\beta + m) - b C_1 (\beta - 2m)(1-3\nu)) \\ + 2b^3 \left(\begin{array}{l} P - \frac{1}{2} a C_2 e^{am} \lambda (\beta + m) \\ - \frac{C_1 e^{am} \lambda (\beta - 2m)(1+3\nu)}{2a^2} \end{array} \right) \end{array} \right) \quad (24)$$

$$C_4 = \frac{e^{-am}}{2\lambda(a^3 - b^3)(1+\nu)} \left(\begin{array}{l} a^4 C_2 e^{am} \lambda (\beta + m) - 2a^3 P + a C_1 e^{am} \lambda (\beta - 2m)(1-3\nu) \\ - b e^{am} \lambda (\beta (C_1 (1-3\nu) + b^3 C_2) + m(b^3 C_2 - 2C_1 (1-3\nu))) \end{array} \right) \quad (25)$$

Applying the same process we obtain the second iteration u_2 .

$$u_2 = C_6 r + \frac{C_5}{r^2} + \frac{C_1}{4} (\beta^2 - 3\beta m + 2m^2) - \frac{C_4}{4} (\beta + m)r^2 + \frac{C_2}{40} (\beta^2 + 3\beta m + 2m^2)r^3 + \frac{C_3 (\beta - 2m)}{2r} \quad (26)$$

By applying the boundary conditions, the integration constants C_5 and C_6 can be calculated as follows

$$C_5 = \frac{a^3 e^{-am}}{80\lambda(a^3 - b^3)(1-2\nu)} \left(\begin{array}{l} 40Pb^3 + 20b^3 C_4 e^{am} \lambda (\beta + m)(b-a) \\ - b^3 C_2 e^{am} \lambda (\beta^2 + 3\beta m + 2m^2)(3-\nu)(b^2 - a^2) \\ - 20b^2 \nu C_1 e^{am} \lambda (\beta^2 - 3\beta m + 2m^2) \left(1 - \frac{b}{a}\right) \\ + 20C_3 b e^{am} \lambda (\beta - 2m)(1-3\nu) \left(1 - \frac{b^2}{a^2}\right) \end{array} \right) \quad (27)$$

$$C_6 = \frac{1}{40\lambda(b^3 - a^3)(1+\nu)} \left(\begin{array}{l} -20a^2 C_1 \nu e^{am} \lambda (\beta^2 - 3\beta m + 2m^2)(b^2 - a^2) \\ C_2 e^{am} \lambda (\beta^2 + 3\beta m + 2m^2)(3-\nu)(b^5 - a^5) \\ + 20a C_3 e^{am} \lambda (\beta - 2m)(1-3\nu)(b-a) \\ + 20C_4 e^{am} \lambda (\beta + m)(b^4 - a^4) + 40a^3 P \end{array} \right) \quad (28)$$

We stop at this point as the remaining iterates are too long to show here. The more the iteration, the more accurate the answer.

Substitution of $u(r) = u_2(r)$ in Eqs. (3-5) gives stress relations as follows:

$$\sigma_r = \frac{\lambda}{40r^3} e^{mr} \left\{ r \begin{pmatrix} 40C_3m(1-3\nu) + \beta^2(C_2r^4(3-\nu) + 20C_1r\nu) \\ -\beta(20C_4r^3 + 20C_3(1-3\nu) + 3mr(20C_1\nu + C_2r^3(\nu-3))) \\ +2r(20C_6r(1+\nu) + m(m(20C_1\nu + C_2r^3(3-\nu)) - 10C_4r^2)) \end{pmatrix} - 80C_5(1-2\nu) \right\} \quad (29)$$

$$\sigma_\theta = \sigma_\phi = \frac{1}{40r^3} e^{mr} \lambda \left\{ 40C_5(1-2\nu) + r \begin{pmatrix} \beta^2(10C_1r + C_2r^4(1+3\nu)) + \beta \begin{pmatrix} -30C_1mr + 20C_3(1-\nu) \\ +r^3 \begin{pmatrix} 3C_2mr(1+3\nu) \\ -10C_4(1+2\nu) \end{pmatrix} \end{pmatrix} \\ +2 \left(20C_3m(-1+\nu) + r \left(10C_1m^2 + r \begin{pmatrix} 20C_6(1+\nu) \\ +mr \begin{pmatrix} C_2mr(1+3\nu) \\ -5C_4(1+2\nu) \end{pmatrix} \end{pmatrix} \right) \right) \end{pmatrix} \right\} \quad (30)$$

3 RESULTS AND DISCUSSION

Consider a thick FGM spherical pressure vessel with the inner radius $a = 1m$ outer radius $b = 1.5m$, with material property of ($n = 2, E_i = 30GPa$). It is assumed that the Poisson's ratio ν is a constant with the value of 0.3. The sphere is assumed to be under the action of a constant pressure $P = 3KPa$. In order to present the numerical results, the following formal dimensionless and normalized variables are used:

$$\begin{cases} \bar{\sigma}_i = \frac{\sigma_i}{-P} & , & \bar{u} = \frac{u}{a} \\ \bar{E} = \frac{E}{E_i} & , & \bar{r} = \frac{r}{a} \end{cases}$$

In this study, in order to do a numerical analysis of the problem, a geometry specimen was modeled using a commercial finite elements code, ANSYS 12, for a comparative study. In the FE model (finite element model), due to symmetry, only a quarter of the spherical specimen geometry is considered (Fig. 2). An 8-node axisymmetric quadrilateral element is used to represent the FGM specimen. For the purpose of modeling the FGM sphere, the variation in material properties was implemented by 20 layers, with each layer having a constant value of material properties. Fig. 2 illustrates the meshing region.

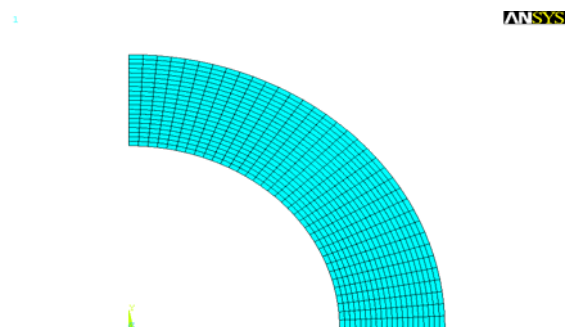


Fig. 2
Finite element mesh region.

Radial displacement obtained from the iterative technique (two and four iterations) and FEM is summarized in Table 1. The results show that the iterative technique gives us reliable accuracy for the problem in hand and the

more the iteration, the more accurate the answer. In Fig. 3, the comparison radial displacement between the two methods is performed. This Figure also acknowledges the desired accuracy.

Radial stress obtained from the iterative technique and FEM are tabloid in Table 2. The errors are small and acceptable, and more iteration can improve the response. The radial, tangential and von Mises stresses, obtained from iterative technique and FEM, are compared in Figs. 4, 5 and 6. In these Figures, it is clearly evident that the iterative technique has high accuracy. \bar{u} against \bar{r} is shown in Fig. 7. This Figure demonstrates that at the constant \bar{r} , by increasing the parameter n , the displacement is decreased, as could be seen. $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ versus \bar{r} for different parameters n are shown in Figs. 8 and 9. These Figures show that the graded material properties have significant effects on the mechanical behavior of the hollow sphere. For different values of n , dimensionless modulus of elasticity along the \bar{r} direction is plotted in Fig. 10.

Table 1
Comparison of radial displacement obtained by the iterative technique and FEM

r	\bar{u}_2 (Iterative Technique)	\bar{u}_4 (Iterative Technique)	\bar{u} (FEM)	Error%
1	0.00000043604	0.00000044394	0.00000044235	0.35
1.1	0.00000034962	0.00000035677	0.00000035536	0.39
1.2	0.00000029807	0.00000030408	0.00000030285	0.40
1.3	0.00000026592	0.00000027076	0.0000002697	0.39
1.4	0.00000024453	0.00000024858	0.00000024764	0.37
1.5	0.00000022925	0.00000023297	0.00000023211	0.36

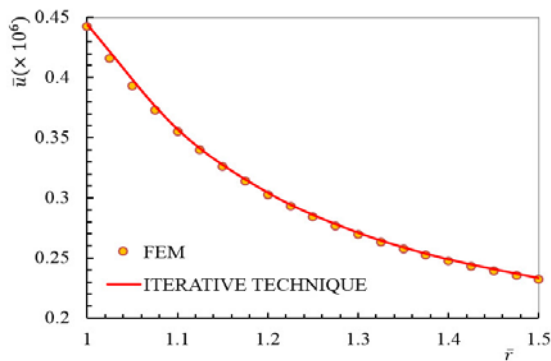


Fig. 3
Comparison of radial displacement obtained by the iterative technique and FEM.

Table 2
Radial displacement and tangential stresses obtained by the iterative technique and FEM.

r	$\bar{\sigma}_r$ (Iterative Technique)	$\bar{\sigma}_r$ (FEM)	Error%
1	1	1	0
1.1	0.7787	0.7783	0.051
1.2	0.584	0.582	0.34
1.3	0.398	0.395	0.75
1.4	0.2073	0.2057	0.77
1.5	0	0	0

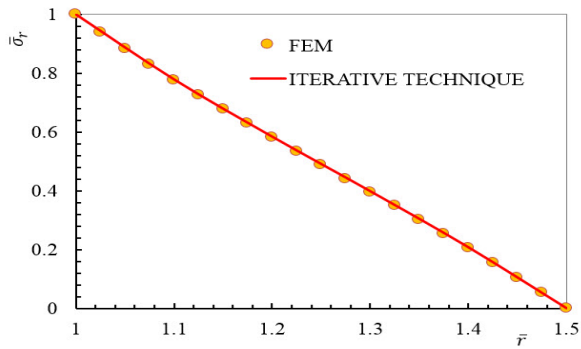


Fig. 4
Comparison of radial stress obtained by the iterative technique and FEM.

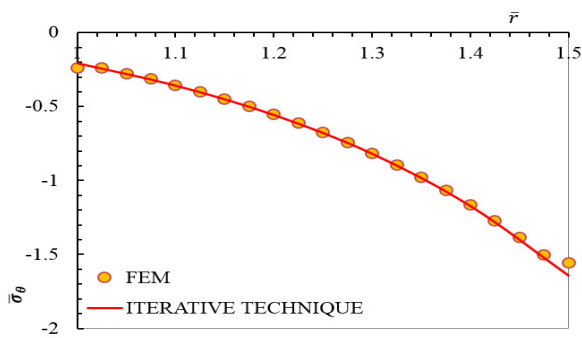


Fig. 5
Comparison of tangential stress obtained by the iterative technique and FEM.

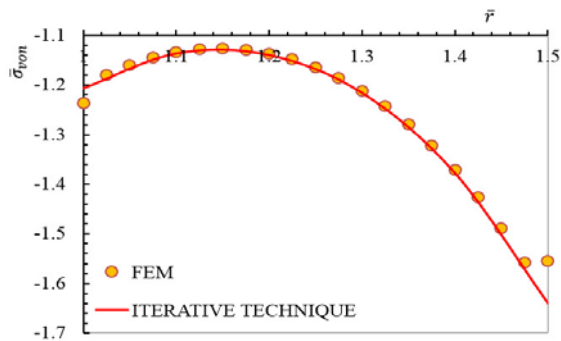


Fig. 6
Comparison of von Mises stress obtained by the iterative technique and FEM.

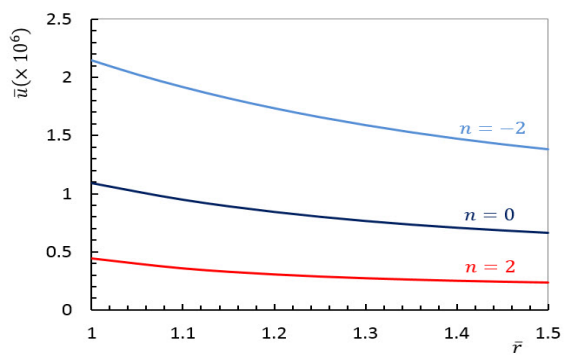


Fig. 7
Distribution of displacement in r direction of the sphere versus to \bar{r} .

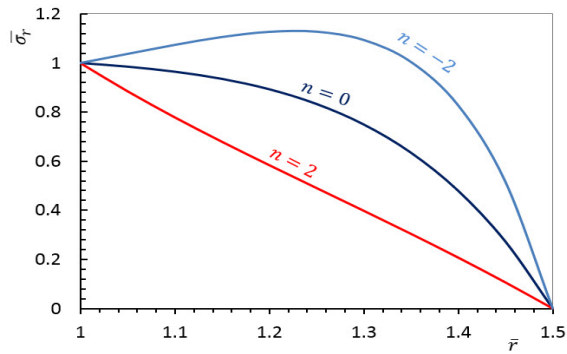


Fig. 8
Distribution of radial stress of the sphere versus to \bar{r} .

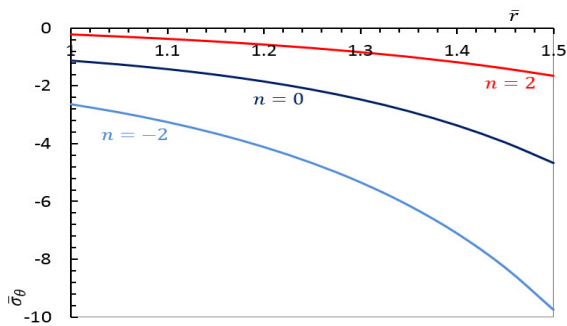


Fig. 9
Distribution of tangential stress of the sphere versus to \bar{r} .

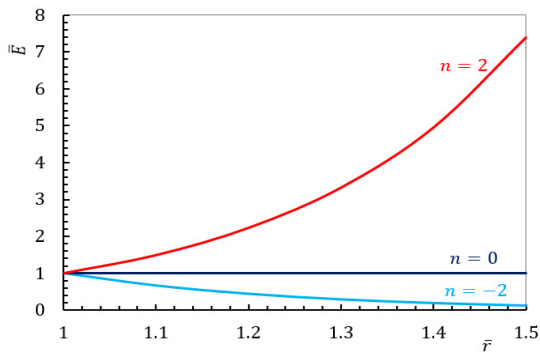


Fig. 10
Distribution of modulus of elasticity.

4 CONCLUSIONS

The present paper has discussed the applicability of iterative technique to obtain the stresses and displacements of FGM thick spherical shells with exponentially-varying properties. A semi-analytical iterative technique is employed to solve the governing Navier equation and numerical solution, using FEM is also presented. The results suggest a good agreement between the semi-analytical method and the results based on FEM. Also to show the effect of inhomogeneity on the stress distributions, different values were considered for material inhomogeneity parameter n . The presented results also show that the material inhomogeneity has a significant influence on the mechanical behaviors of the spherical shells with exponentially-varying properties.

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