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Exact 3-D Solution for Free Bending Vibration of Thick FG Plates and Homogeneous Plate Coated by a Single FG Layer on Elastic Foundations

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ABSTRACT

This paper presents new exact 3-D (three-dimensional) elasticity closed-form solutions for out-of-plane free vibration of thick rectangular single layered FG (functionally graded) plates and thick rectangular homogeneous plate coated by a functionally graded layer with simply supported boundary conditions. It is assumed that the plate is on a Winkler-Pasternak elastic foundation and elasticity modulus and mass density of the FG layer vary exponentially through the thickness of the FG layer, whereas Poisson's ratio is constant. In order to solve the equations of motion, a proposed displacement field is used for each layer. Influences of stiffness of the foundation, inhomogeneity of the FG layer and coating thickness-to-total thickness ratio on the natural frequencies of the plates are discussed. Numerical results presented in this paper can serve as benchmarks for future vibration analyses of single layered FG plates and coated plates on elastic foundations.

Keywords: Free bending vibration; Exact 3-D solution; Thick FG plates; Homogeneous plate coated by a single FG layer; Winkler-Pasternak elastic foundation

1 INTRODUCTION

FUNCTIONALLY graded materials are a new class of heterogeneous composite structures first introduced by a group of Japanese scientists in 1984 to address the needs of aggressive thermal environment [1, 2]. Typically, FGMs are made of a ceramic and a metal in such a way that the ceramic resists the rigorous thermal shocks from the high temperature environment, whereas the metal is applied to reduce the large tensile stresses occurring on the ceramic surface and causes large-loading capacity. The material properties of FGMs vary continuously and smoothly from one interface to the other. This important property is achieved by gradually varying the volume fraction of material components and results continuous and smooth behavior for stress distribution and eliminates ruptured stress distribution in the usual laminate composite structures.

According to a comprehensive survey of the literature, it is found that over the years, free vibration of single layered FG plates and coated plates has been a focus of numerous studies according to various 2-D and 3-D plate theories, but few researches are based on 3-D elasticity theory. The 3-D elasticity theory is able and accurate for analysis of thick FG plates and thick coated plates compared to the other theories of plate. Tarn and Wang [3, 4] obtained 3-D elasticity thermal deformations of FG plates by using an asymptotic expansion method. Chen and Ding

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[5] investigated bending of FG piezoelectric plates using state-space method. Also, Chen et al. [6] applied state-space method to study free vibration of FG magneto-electro-elastic plates. Reddy and Cheng [7] used asymptotic expansion method to analysis thermo-mechanical behavior of simply supported FG plates. They studied the effects of the exponent of the volume fraction law on the structural response under thermal and/or mechanical loads. Vel and Batra [8] studied the large deformations of FG thick rectangular plates with simply supported boundary conditions subjected to transversely mechanical and thermal loading on its top and/or bottom surfaces. Also, Vel and Batra [9-10] presented an exact semi-analytical method to investigate thermal deformation, transient thermal stress distribution and free vibration of thick simply supported rectangular FG plates using suitable displacement functions. Zhong and Shang [11] developed a three-dimensional analysis for thermal stresses of a rectangular plate made of orthotropic functionally graded piezoelectric material. They assumed that the plate is simply supported and grounded along its four edges, and mechanical and electric properties of the material vary exponentially along the thickness direction. Zhong and Yu [12] presented an analytical approach for free and force vibration of FG piezoelectric simply supported rectangular plates with exponential variation of material properties along the thickness direction. A three-dimensional elasticity analysis of deformation of a functionally graded coating/substrate structure was carried out by Kashtalyan and Menshykova [13]. They assumed that the coating was a single layered with an exponential variation of material properties through the thickness. Later, they studied the bending of coated plates with FG interlayer [14]. Both of their studies were carried out on the basis of a 3-D elasticity solution for bending response of full simply supported FG plates that previously was developed by Kashtalyan [15]. Huang et al. [16] described an excellent solution for bending of simply supported FG plates resting on Winkler–Pasternak elastic foundations. They used state-space method and assumed that the plate is isotropic at any point, while material properties vary exponentially along the thickness direction. Lu et al. [17] employed a developed hybrid analysis for investigating the bending and free vibration of thick laminated composite rectangular plates. Also, Lu et al. [18] provided an exact solution for free vibration of simply supported rectangular FG plates on Winkler-Pasternak elastic foundation. Li et al. [19, 20] investigated the free vibration of FG sandwich plates and FG plates in thermal environments by using Ritz method and Chebyshev polynomials. Also, Amini et al. [21] utilized Ritz method and Chebyshev polynomials to study three-dimensional free vibration of rectangular FG plates with arbitrary boundary conditions while the plate is resting on an Winkler elastic foundation. Alibeigloo [22] analyzed three-dimensional thermo-elastic behavior of simply supported rectangular FG plates subjected to thermo-mechanical loads. Hosseini-Hashemi et al. [23, 24] presented exact closed-form solutions for free vibration of rectangular FG plates and homogeneous plates coated by a FG layer based on 3-D elasticity theory.

The purpose of this paper is to propose a new exact 3-D elasticity closed-form solutions for out-of-plane free vibration of thick simply supported rectangular single layered FG plates and homogeneous plates coated by a FG layer on Winkler-Pasternak elastic foundations. It is assumed that the elasticity modulus and mass density vary exponentially through the thickness of the FG layer, whereas the Poisson's ratio is constant. To solve the elasticity equations of motion, a proposed displacement field is used for each layer. These displacement fields satisfy boundary conditions on the edges of the plate and substituting of them into 3-D elasticity equations of motion yield some independent equations that can be solved explicitly. A new solution procedure which is more simple than the solution procedures of Hosseini-Hashemi et al. [23, 24] is presented to solve the ordinary differential equations. The rest of solution is completed by satisfying boundary conditions of interface and surfaces of the structure. In order to prove the stability and high accuracy of the present solution, some comparative results are carried out with existing data in the literature. Effects of the foundation stiffness, inhomogeneity of the FG layer, and coating thickness-to-total thickness ratio on the natural frequencies of FG plates and coated plates are investigated and discussed. The present results will be a useful benchmark for evaluating the accuracy of future vibration analyses of FG plates and coated plates on elastic foundations.

2 PROBLEM FORMULATION

Consider a thick rectangular coated plate on a Winkler–Pasternak elastic foundation (Fig. 1) which has the length of a, width of b, total thickness of h and FG coating thickness h_1. The boundary conditions are assumed to be simply supported at all edges of the plate. Cartesian coordinate system (x, y, z) is considered to extract mathematical formulations when x and y axes are located in the bottom plane of the coated plate.

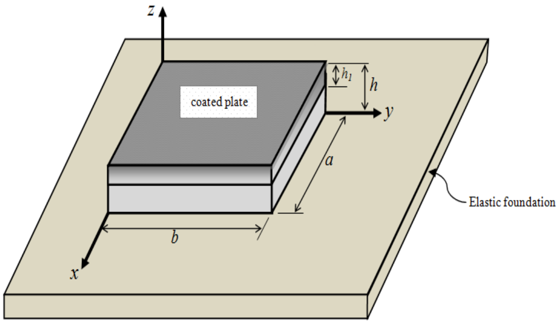


Fig. 1
A rectangular coated plate resting on an elastic foundation, with coordinate system.

Here, it is assumed that the material properties of the FG layer vary in the exponential law through the thickness of the FG layer as:

$$\mu_1 = \mu_0 \exp(\phi(z-(h-h_1))), \qquad \rho_1 = \rho_0 \exp(\phi(z-(h-h_1))) \tag{1}$$

where μ_1 is the gradient index of the material properties of the FG layer, and μ_0, ρ_0 are shear modulus and mass density of the homogeneous substrate, respectively. Poisson's ratio is assumed to be constant and is taken as 0.3 throughout the analysis. It should be emphasized that the elastic constants and the density are assumed to have same gradient index.

In the absence of body forces, the linear 3-D elasticity equations of motion in terms of displacements for an isotropic FG material with the shear modulus $\mu = \mu(z)$, mass density $\rho = \rho(z)$ and constant Poisson's ratio ν in terms of displacement components have the form

$$\mu(z)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{\mu(z)}{1-2\nu}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)+\mu'(z)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\rho(z)\frac{\partial^2 u}{\partial t^2} \tag{2a}$$

$$\mu(z)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\mu(z)}{1-2\nu}\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right)+\mu'(z)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)=\rho(z)\frac{\partial^2 v}{\partial t^2} \tag{2b}$$

$$\mu(z)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\mu(z)}{1-2\nu}\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{2\nu\,\mu'(z)}{1-2\nu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu'(z)\frac{\partial w}{\partial z}=\rho(z)\frac{\partial^2 w}{\partial t^2} \tag{2c}$$

where u, v and w are the components of the displacement vector along the three Cartesian axes and t denotes the time variable. The 3-D elasticity equations have the following simple forms for the homogeneous substrate as:

$$\mu_0\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{\mu_0}{(1-2\nu)}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)=\rho_0\frac{\partial^2 u}{\partial t^2} \tag{3a}$$

$$\mu_0\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\mu_0}{(1-2\nu)}\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right)=\rho_0\frac{\partial^2 v}{\partial t^2} \tag{3b}$$

$$\mu_0\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\mu_0}{(1-2\nu)}\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}\right)=\rho_0\frac{\partial^2 w}{\partial t^2} \tag{3c}$$

In order to solve the Eqs. (2) and (3), the following displacement fields are employed

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \begin{Bmatrix} -g_0(z)\alpha_m \cos(\alpha_m x)\sin(\beta_n y) \\ -g_0(z)\beta_n \sin(\alpha_m x)\cos(\beta_n y) \\ f_0(z)\sin(\alpha_m x)\sin(\beta_n y) \end{Bmatrix} e^{i\omega t}, \quad i=\sqrt{-1} \tag{4a}$$

$$\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \begin{Bmatrix} -g_1(z)\alpha_m \cos(\alpha_m x)\sin(\beta_n y) \\ -g_1(z)\beta_n \sin(\alpha_m x)\cos(\beta_n y) \\ f_1(z)\sin(\alpha_m x)\sin(\beta_n y) \end{Bmatrix} e^{i\omega t}, \quad i=\sqrt{-1} \tag{4b}$$

where $\alpha_m = m\pi/a$, $\beta_n = n\pi/b$ and subscripts 0 and 1 refer to the homogeneous substrate and FG layer, respectively. The above displacement fields are Levinson's representation form that applied to analysis bending and out-of-plane free vibration of isotropic homogeneous rectangular plates [25-27]. $g_i(z)$ and $f_i(z)$ $(i=0,1)$ are four unknown functions. m and n are integers and represent the number of half-waves in x and y directions. The above displacement fields satisfy simply supported boundary conditions which are defined by following equations

$$w_0\big|_{x=0,a} = v_0\big|_{x=0,a} = \sigma_{xx0}\big|_{x=0,a} = 0 \tag{5a}$$

$$w_1\big|_{x=0,a} = v_1\big|_{x=0,a} = \sigma_{xx1}\big|_{x=0,a} = 0 \tag{5b}$$

$$w_0\big|_{y=0,b} = u_0\big|_{y=0,b} = \sigma_{yy0}\big|_{y=0,b} = 0 \tag{5c}$$

$$w_1\big|_{y=0,b} = u_1\big|_{y=0,b} = \sigma_{yy1}\big|_{y=0,b} = 0 \tag{5d}$$

Substitution of the assumed displacement field (4a) into the elasticity Eqs. (3) and substitution of the assumed displacement field (4b) and material properties Eqs. (1) into the elasticity Eqs. (2) and simplifying the results lead to just four different ordinary differential equations as:

$$A_{10}g_0''(z) + A_{20}g_0(z) + A_{30}f_0'(z) = 0 \tag{6a}$$

$$B_{10}f_0''(z) + B_{20}f_0(z) + B_{30}g_0'(z) = 0 \tag{6b}$$

$$A_{11}g_1''(z) + A_{20}g_1'(z) + A_{31}g_1(z) + A_{41}f_1'(z) + A_{51}f_1(z) = 0 \tag{6c}$$

$$B_{11}f_1''(z) + B_{21}f_1'(z) + B_{31}f_1(z) + B_{41}g_1'(z) + B_{51}g_1(z) = 0 \tag{6d}$$

where the constant coefficients A_{ij} and B_{ij} ($i=1,...,5$ and $j=0,1$) are given in Appendix.

Eqs .(6) can be simplified by using the derivative operator D as follows

$$\left[A_{10}D^2 + A_{20}\right]g_0(z) + \left[A_{30}D\right]f_0(z) = 0 \tag{7a}$$

$$\left[B_{10}D^2 + B_{20}\right]f_0(z) + \left[B_{30}D\right]g_0(z) = 0 \tag{7b}$$

$$\left[A_{11}D^2 + A_{21}D + A_{31}\right]g_1(z) + \left[A_{41}D + A_{51}\right]f_1(z) = 0 \tag{7c}$$

$$\left[B_{11}D^2 + B_{21}D + B_{31}\right]f_1(z) + \left[B_{41}D + B_{51}\right]g_1(z) = 0 \tag{7d}$$

The algebraic solution of the set of Eqs. (7) leads to four ordinary equations. So, the functions $g_i(z)$ and $f_i(z)$ $(i=0,1)$ are obtained easily from Eqs. (7) as follows

$$g_0(z) = L_{10}\exp(s_{10}z) + L_{20}\exp(s_{20}z) + L_{30}\exp(s_{30}z) + L_{40}\exp(s_{40}z) \tag{8a}$$

$$f_0(z) = R_{10}\exp(s_{10}z) + R_{20}\exp(s_{20}z) + R_{30}\exp(s_{30}z) + R_{40}\exp(s_{40}z) \tag{8b}$$

$$g_1(z) = L_{11}\exp(s_{11}z) + L_{21}\exp(s_{21}z) + L_{31}\exp(s_{31}z) + L_{41}\exp(s_{41}z) \tag{8c}$$

$$f_1(z) = R_{11}\exp(s_{11}z) + R_{21}\exp(s_{21}z) + R_{31}\exp(s_{31}z) + R_{41}\exp(s_{41}z) \tag{8d}$$

where the coefficients s_{ij} ($i=1,2,3,4$ and $j=0,1$) are explicitly expressed as follows

$$\begin{Bmatrix} s_{10} \\ s_{20} \\ s_{30} \\ s_{40} \end{Bmatrix} = \pm\frac{1}{2}\sqrt{\frac{\mu_0(1-\nu)(4\lambda_{mn}^2)+(4\nu-3)\rho_0\omega^2 \pm \rho_0\omega^2}{\mu_0(1-\nu)}} \tag{9a}$$

$$\begin{Bmatrix} s_{11} \\ s_{21} \\ s_{31} \\ s_{41} \end{Bmatrix} = -\frac{1}{2}\left(\varphi \pm \sqrt{\frac{\mu_0(1-\nu)(\phi^2+4\lambda_{mn}^2)+(4\nu-3)\rho_0\omega^2 \pm \sqrt{\rho_0^2\omega^4-16\phi^2\lambda_{mn}^2\mu_0^2(1-\nu)\nu}}{\mu_0(1-\nu)}}\right) \tag{9b}$$

where

$$\lambda_{mn}^2 = \alpha_m^2 + \beta_n^2 \tag{10}$$

The coefficients R_{ij} ($i=1,2,3,4$ and $j=0,1$) can be written in terms of L_{ij}. Substituting $g_0(z)$ and $f_0(z)$ into either Eqs. (7a) or (7b) and Substituting $g_1(z)$ and $f_1(z)$ into either Eqs. (7c) or (7d) yield the following 8 dependency equations

$$R_{10} = -\frac{A_{10}s_{10}^2 + A_{20}}{A_{30}s_{10}}L_{10} \tag{11a}$$

$$R_{20} = -\frac{A_{10}s_{20}^2 + A_{20}}{A_{30}s_{20}}L_{20} \tag{11b}$$

$$R_{30} = -\frac{A_{10}s_{30}^2 + A_{20}}{A_{30}s_{30}}L_{30} \tag{11c}$$

$$R_{40} = -\frac{A_{10}s_{40}^2 + A_{20}}{A_{30}s_{40}}L_{40} \tag{11d}$$

and

$$R_{11} = -\frac{A_{11}s_{11}^2 + A_{21}s_{11} + A_{31}}{A_{41}s_{11} + A_{51}}L_{11} \tag{12a}$$

$$R_{21} = -\frac{A_{11}s_{21}^2 + A_{21}s_{21} + A_{31}}{A_{41}s_{21} + A_{51}}L_{21} \tag{12b}$$

$$R_{31} = -\frac{A_{11}s_{31}^2 + A_{21}s_{31} + A_{31}}{A_{41}s_{31} + A_{51}}L_{31} \tag{12c}$$

$$R_{41} = -\frac{A_{11}s_{41}^2 + A_{21}s_{41} + A_{31}}{A_{41}s_{41} + A_{51}}L_{41} \tag{12d}$$

Satisfying the boundary conditions of interface and surfaces of the plate is the final step to extract the natural frequencies related to out-of-plane modes. The continuity conditions of displacements and stresses between the FG coating layer and the homogeneous substrate are defined by

$$\sigma_{zz0} = \sigma_{zz1}, \quad \sigma_{xz0} = \sigma_{xz1}, \quad \sigma_{yz0} = \sigma_{yz1} \qquad (z = h - h_1) \tag{13a}$$

$$u_0 = u_1, \quad v_0 = v_1, \quad w_0 = w_1 \qquad (z = h - h_1) \tag{13b}$$

The top surface of the plate is free, while the bottom surface is resting on a Winkler-Pasternak elastic foundation. The boundary conditions related to these surfaces are defined by

$$\sigma_{zz1} = 0 \qquad (z = h) \tag{14a}$$

$$\sigma_{xz1} = \sigma_{yz1} = 0 \qquad (z = h) \tag{14b}$$

$$\sigma_z = k_w\, w - k_{px} \frac{\partial^2 w}{\partial x^2} - k_{py} \frac{\partial^2 w}{\partial y^2} \qquad (z = 0) \tag{14c}$$

$$\sigma_{xz0} = \sigma_{yz0} = 0 \qquad (z = 0) \tag{14d}$$

Eq. (14c) expressed the reaction–deflection relation between the bottom surface of the plate and Winkler-Pasternak elastic foundation. k_w is the Winkler stiffness of foundation. k_{px} and k_{px} are shear stiffness related to the x and y axes, respectively. It is true that $k_{px} = k_{py} = k_p$ for an isotropic foundation. By replacing the displacement fields (4a) and (4b) into Eqs. (13) and (14) and simplifying the results, 16 continuity condition equations and boundary condition equations reduce to only 8 independent equations as follows

$$g_0 = g_1, \quad f_0 = f_1 \qquad (z = h - h_1) \tag{15a}$$

$$g_0' = g_1', \quad f_0' = f_1' \qquad (z = h - h_1) \tag{15b}$$

$$(1-\nu) f_1'(z) + \nu\left(\alpha_m^2 + \beta_n^2\right) g_1(z) = 0 \qquad (z = h) \tag{15c}$$

$$f_1(z) - g_1'(z) = 0 \qquad (z = h) \tag{15d}$$

$$\frac{2\mu_0}{1-2\nu}\Big((1-\nu) f_0'(z) + \nu\left(\alpha_m^2 + \beta_n^2\right) g_0(z)\Big) - \Big(k_w + k_p(\alpha_m^2 + \beta_n^2)\Big) f_0(z) = 0 \qquad (z = 0) \tag{15e}$$

$$f_0(z) - g_0'(z) = 0 \qquad (z = 0) \tag{15f}$$

Eqs. (15) are 8 algebraic equations in terms of unknown coefficients L_{ij} ($i = 1,2,3,4$ and $j = 0,1$). For a nontrivial solution, the determinant of coefficient matrix of equations must be zero. By solving the consequence characteristic equation, the exact natural frequencies related to out-of-plane vibration modes are attained.

The procedure of extracting out-of-plane natural frequencies of a single layered FG plate is similar to that of coated plates. In order to obtain frequencies of a single layered FG plate, only one of the displacement fields (4) is used and replaced to elasticity Eqs. (2) and surface boundary conditions (14). The rest of solution procedure is similar to that of coated plate.

4 RESULTS AND DISCUTIONS

The numerical results are presented using previously 3-D elasticity closed-form exact solution. In order to validate accuracy and stability of the developed solution and investigate different mechanical conditions such as stiffness of the foundation, inhomogeneity of the FG layer, and coating thickness-to-total thickness ratio on the natural frequencies, some numerical results are presented that some of them are compared with corresponding results of Lu et al. [18]. Without losing generality, numerical results are presented for square plates on isotropic foundations. In all numerical results, dimensionless forms of natural frequency and foundation stiffness are used as follows

$$\bar{\omega} = \omega a^2 \sqrt{\rho/E}\,/h \tag{16a}$$

$$K_W = k_w a^4 / D, \quad K_P = k_p a^2 / D \tag{16b}$$

where $D = \mu_0 h^3 / 6(1-\nu)$ is a reference of bending rigidity of the plate. In Table 1 and Table 2 , the first two dimensionless out-of-plane natural frequencies of a square single layered FG plate related to nine pairs of the three minimum positive values of integers m and n for the length-to-thickness ratios of 10 and 5 are tabulated, respectively. Excellent agreement can be observed from the comparative results of these tables. It is important that softer surface (case1) or harder surface (case2) of the plate is attached to the elastic foundation, because the results of each case is different. The results of each case are presented for three different values of dimensionless foundation stiffness, while the gradient index ϕ is assumed to be $1/h\ln 10$. It is obvious that the results of case 2 are more than the counterpoint results of case 1. Similar results are presented for square coated plates in Table 3. The foundation is assumed to be attached to the homogeneous substrate and the coating thickness-to-total thickness ratio h_1/h of 1/4 is considered. A glance at the results in Tables 1,2 and 3 reveals that increasing the stiffness of foundation increases the dimensionless natural frequencies of single layered plates and coated plates. The first dimensionless natural frequency related to the each integer pair (m, n) increases remarkably by increasing the foundation stiffness, while the second frequency don't vary sensibly by increasing the foundation stiffness.

Variations of the two first dimensionless natural frequencies of square single layered square FG plates versus the shear elasticity modulus ratio $\mu_1(h)/\mu_0$ for length-to-thickness ratios of 10 (moderately thick plates) and 5(thick plates) are depicted in Figs. 2 and 3, respectively. The ratio $\mu_1(h)/\mu_0$ gives the slope of variation of material properties through the thickness of the FG layer. The curves are depicted for three different values of foundation stiffness $K_w = K_p = 0$ (solid line), $K_w = 100, K_p = 0$ (dash line) and $K_w = 100, K_p = 10$ (dotted line).

Table1
Dimensionless out-of-plane natural frequency of a square single layered FG plate ($a/h = 10$) for three values of foundation stiffness

Method	mode (m, n)	$\phi = 1/h\ln 10$ Case 1 $K_W = 0$ $K_P = 0$	$K_W = 100$ $K_P = 0$	$K_W = 100$ $K_P = 10$	$\phi = 1/h\ln 10$ Case 2 $K_W = 0$ $K_P = 0$	$K_W = 100$ $K_P = 0$	$K_W = 100$ $K_P = 10$
Present	1st	5.12953	5.34360	5.74254	5.12953	5.34984	5.76001
[18]	(1,1)	5.1295	-	5.7425	5.1295	5.3498	5.7600
Present	2nd	12.3204	12.4059	12.8191	12.3204	12.4121	12.8552
[18]	(1,2)/(2,1)	12.320	-	12.819	-	-	-
Present	3rd	19.0098	19.0625	19.4732	19.0098	19.0686	19.5267
[18]	(2,2)	19.010	-	19.473	-	-	-
Present	4th	23.2341	23.2759	23.6839	23.2341	23.2819	23.7485
[18]	(1,3)/(3,1)	23.234	-	23.684	-	-	-
Present	5th	29.2671	29.2989	29.7027	29.2671	29.3048	29.7835
[18]	(2,3)/(3,2)	-	-	-	-	-	-
Present	6th	38.6371	38.6597	39.0569	38.6371	38.6654	39.1637
[18]	(3,3)	-	-	-	-	-	-
Present	7th	46.5183	46.5188	46.5196	46.5183	46.5184	46.5186
[18]	(1,1)	46.518	-	46.520	-	-	-
Present	8th	73.4155	73.4162	73.4196	73.4155	73.4157	73.4164
[18]	(1,2)/(2,1)	73.415	-	73.420	-	-	-
Present	9th	92.6846	92.6855	92.6927	92.6846	92.6848	92.6864
[18]	(2,2)	92.685	-	92.693	-	-	-
Present	10th	103.486	103.487	103.498	103.486	103.487	103.489
[18]	(1,3)/(3,1)	103.49	-	103.50	-	-	-
Present	11th	117.748	117.749	117.765	117.748	117.748	117.752
[18]	(2,3)/(3,2)	-	-	-	-	-	-
Present	12th	138.048	138.050	138.077	138.048	138.049	138.055
[18]	(3,3)	-	-	-	-	-	-

It can be concluded from these Figures that the two first dimensionless natural frequencies of a single layered FG plate decrease when the gradient index increases. It is deduced from Figs. 2 and 3 that the slope of curves decrease by increasing the value of gradient index. Also, it is evident that the natural frequency is higher when the foundation is harder and the effects of elastic foundation on the first natural frequencies is more significant when compared to second natural frequency.

In Figs. 4(a) and 4(b), the variation of first dimensionless natural frequency of single layered FG plates versus the aspect ratio b/a for length-to-thickness ratios of 10 and 5 are depicted, respectively. The shear elasticity modulus ratio $\mu_1(h)/\mu_0$ is assumed to be taken as 10. The curves of Figs. 4(a) and 4(b) are depicted for three different values of foundation stiffness as Figs. 2 and 3. It can be observed from Figs. 4(a) and 4(b) that the effect of elastic foundation on the first dimensionless frequency increases as the aspect ratio b/a is enhanced.

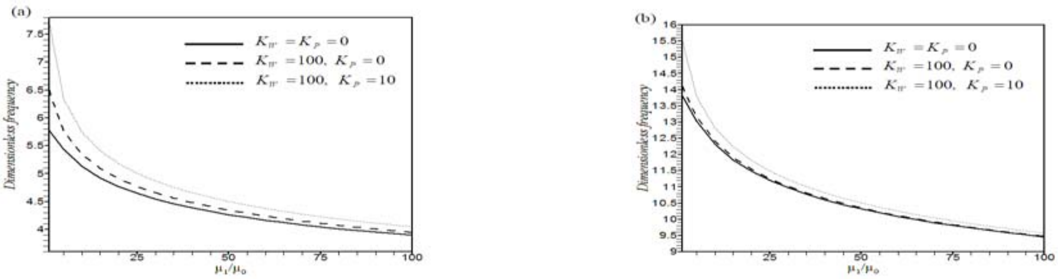
Figs. 5(a) and 5(b) show variation of two first dimensionless frequencies related to integer pair (1, 1) of single layered square FG plates versus foundation stiffness. It is assumed that Winkler stiffness and shear stiffness are equal ($K = K_w = K_P$) and the ratio $\mu_1(h)/\mu_0$ is 10. These Figures illustrated that the foundation influences the first frequency more significant than the second frequency. Also, the slope of curves decrease as the foundation stiffness K is increased.

Table 2
Dimensionless out-of-plane natural frequency of a square single layered FG plate ($a/h = 5$) for three values of foundation stiffness

Method	mode (m, n)	$\phi = 1/h \ln 10$ Case 1			$\phi = 1/h \ln 10$ Case 2		
		$K_W = 0$, $K_P = 0$	$K_W = 100$, $K_P = 0$	$K_W = 100$, $K_P = 10$	$K_W = 0$, $K_P = 0$	$K_W = 100$, $K_P = 0$	$K_W = 100$, $K_P = 10$
Present	1st	4.75245	4.95855	5.33824	4.75245	4.98241	5.40721
[18]	(1,1)	4.7524	-	5.3382	4.7524	4.9824	5.4072
Present	2nd	10.5449	10.6251	11.0043	10.5449	10.6477	11.1396
[18]	(1,2)/(2,1)	10.545	-	11.004	-	-	-
Present	3rd	15.3776	15.4262	15.7929	15.3776	15.4477	15.9887
[18]	(2,2)	15.378	-	15.793	-	-	-
Present	4th	18.2413	18.2795	18.6379	18.2413	18.3005	18.8724
[18]	(1,3)/(3,1)	18.241	-	18.638	-	-	-
Present	5th	22.1433	22.1720	22.5181	22.1433	22.1924	22.8095
[18]	(2,3)/(3,2)	-	-	-	-	-	-
Present	6th	23.1711	23.1748	23.1820	23.1711	23.1720	23.1736
[18]	(1,1)	23.171	-	23.182	-	-	-
Present	7th	27.8752	27.8950	28.2216	27.8752	27.9149	28.6055
[18]	(3,3)	-	-	-	-	-	-
Present	8th	36.3231	36.3297	36.3618	36.3231	36.3245	36.3316
[18]	(1,2)/(2,1)	36.323	-	36.362	-	-	-
Present	9th	45.4793	45.4889	45.5630	45.4793	45.4813	45.4971
[18]	(2,2)	45.479	-	45.563	-	-	-
Present	10th	50.4501	50.4620	50.5755	50.4501	50.4524	50.4759
[18]	(1,3)/(3,1)	50.450	-	50.576	-	-	-
Present	11th	56.7477	56.7636	56.9577	56.7477	56.7506	56.7884
[18]	(2,3)/(3,2)	-	-	-	-	-	-
Present	12th	64.9480	64.9719	65.3651	64.9480	64.9518	65.0197
[18]	(3,3)	-	-	-	-	-	-

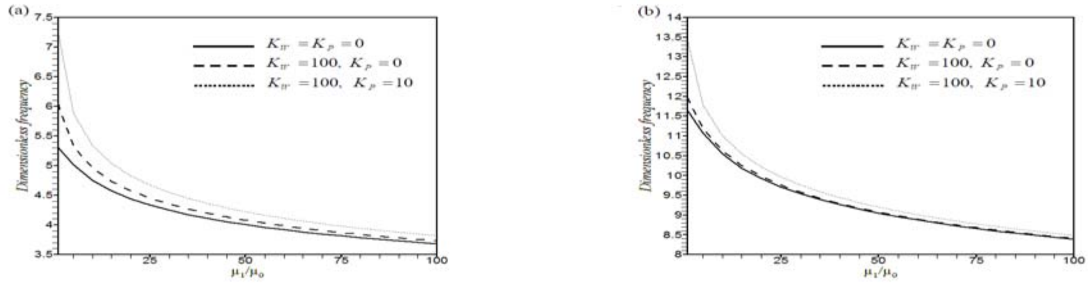
Table 3
Dimensionless out-of-plane natural frequency of a square coated plates for $\phi = 1/h_1 \ln 10, h1/h = 1/4$ and three values of foundation stiffness

Method	mode (m, n)	$a/h = 10$			$a/h = 5$		
		$K_W = 0$ $K_P = 0$	$K_W = 100$ $K_P = 0$	$K_W = 100$ $K_P = 10$	$K_W = 0$ $K_P = 0$	$K_W = 100$ $K_P = 0$	$K_W = 100$ $K_P = 10$
Present	1st (1,1)	6.06783	6.47182	7.20225	5.35296	5.76377	6.48550
Present	2nd (1,2)/(2,1)	14.1818	14.3501	15.1510	11.2119	11.3901	12.1992
Present	3rd (2,2)	21.4119	21.5189	22.3430	15.7828	15.8983	16.7308
Present	4th (1,3)/(3,1)	25.8503	25.9368	26.7721	18.3964	18.4900	19.3243
Present	5th (2,3)/(3,2)	32.0462	32.1137	32.9629	21.8665	21.9387	22.7613
Present	6th (3,3)	41.3903	41.4401	42.3084	26.8034	26.8542	27.6288
Present	7th (1,1)	46.4913	46.4923	46.4943	23.1067	23.1160	23.1345
Present	8th (1,2)/(2,1)	73.2989	73.3006	73.3090	35.9571	35.9773	36.0747
Present	9th (2,2)	92.4269	92.4292	92.4476	44.4080	44.4448	44.7200
Present	10th (1,3)/(3,1)	103.104	103.107	103.134	48.5629	48.6149	49.0933
Present	11th (2,3)/(3,2)	117.128	117.132	117.175	53.0322	53.1096	54.0266
Present	14th (3,3)	136.873	136.877	136.958	57.3176	57.4245	59.1802



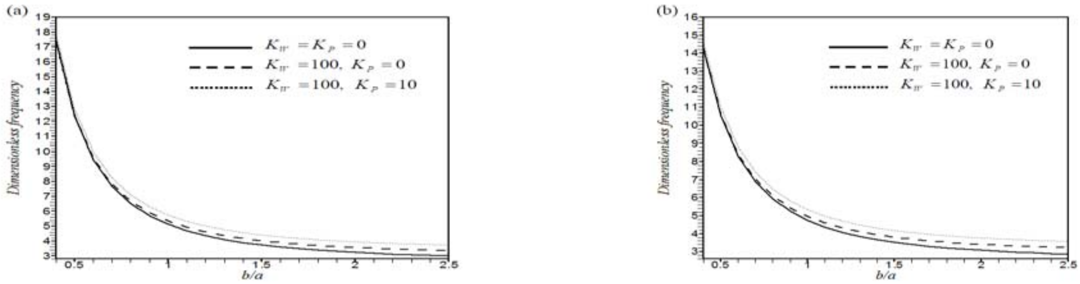
(a) (b)

Fig. 2
Two first out-of-plane dimensionless natural frequency of square single layered FG plate ($a/h = 10$) versus shear elasticity modulus ratio $\mu_1(h)/\mu_0$: a) (m, n)= (1,1); b) (m, n)= (1,2)/(2,1).



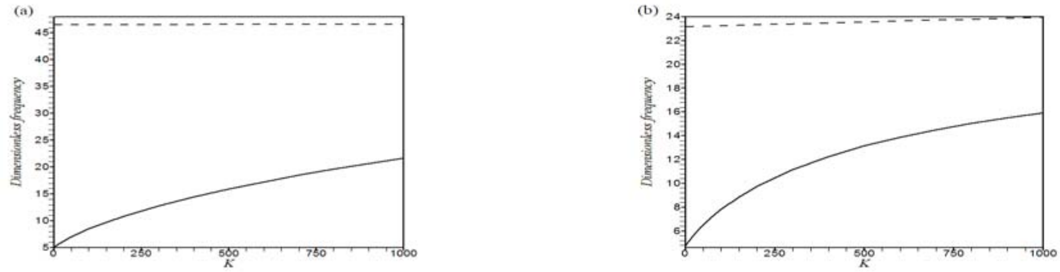
(a) (b)

Fig. 3
Two first out-of-plane dimensionless natural frequency of square single layered FG plate ($a/h = 5$) versus shear elasticity modulus ratio $\mu_1(h)/\mu_0$: a) (m, n)= (1,1); b) (m, n)= (1,2)/(2,1).



(a) (b)

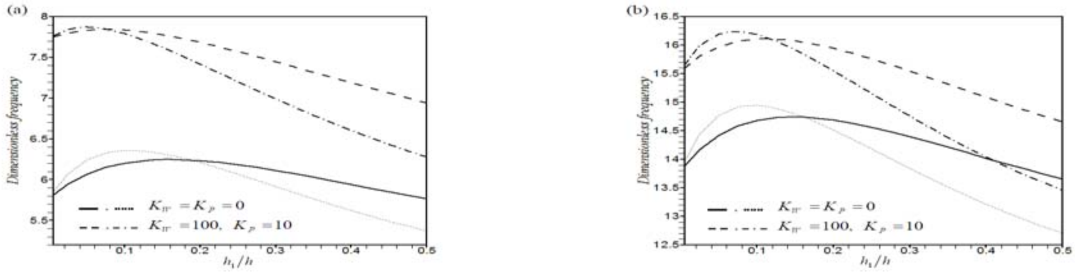
Fig. 4
First out-of-plane dimensionless natural frequency of single layered FG plates versus width-to-length ratio b/a : a) length-to-thickness of 10; b) length-to-thickness of 5.



(a) (b)

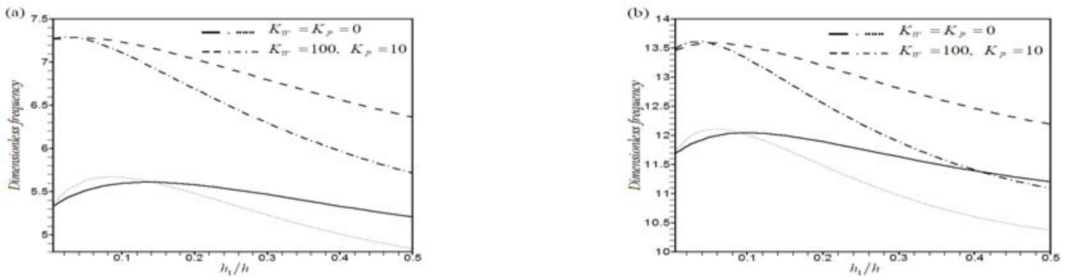
Fig. 5
Two first out-of-plane dimensionless natural frequencies of square single layered FG plates related to the integer pair (1,1) versus stiffness K for: a) length-to-thickness of 10; b) length-to-thickness of 5.

In order to investigate the effect of coating thickness-to-total thickness ratio h_1/h, stiffness of foundation and gradient index of coating layer on the two first dimensionless natural frequencies of square coated plate, attention is focused on Figs. 6 and 7 which are depicted for length-to-thickness ratios of 10 (moderately thick) and 5 (thick), respectively. For sufficiently small values of h_1/h, the first two dimensionless natural frequencies increases when the coating thickness-to-total thickness ratio h_1/h increases, while it is not true for higher values of coating thickness-to-total thickness ratio h_1/h. So, it is clear that for single values of coating thickness-to-total thickness ratio h_1/h, two first dimensionless frequencies should be maximum. It can be concluded that the variation of two first dimensionless natural frequencies versus the foundation stiffness are more significant for smaller values of gradient index and coating thickness-to-total thickness ratio h_1/h. Also, it is deduced that increasing the value of foundation stiffness decreases the ascendant part of curves.



(a) (b)

Fig. 6
Two first out-of-plane dimensionless natural frequencies of a square coated plate ($a/h = 10$) versus h_1/h for $\mu_1(h)/\mu_0 = 5$ (solid line and dashed line) and $\mu_1(h)/\mu_0 = 10$ (dotted line and dash-dotted line): a) (*m*, *n*)= (1,1); b) (*m*, *n*)= (1,2)/(2,1).



(a) (b)

Fig. 7
Two first out-of-plane dimensionless natural frequencies of a square coated plate ($a/h = 5$) versus h_1/h for $\mu_1(h)/\mu_0 = 5$ (solid line and dashed line) and $\mu_1(h)/\mu_0 = 10$ (dotted line and dash-dotted line): a) (*m*, *n*)= (1,1); b) (*m*, *n*)= (1,2)/(2,1).

4 CONCLUSIONS

This paper presented new exact closed-form 3-D elasticity solutions for the out-of-plane free vibration of thick single layered FG plates and coated plates resting on Winkler–Pasternak elastic foundations. The material properties were assumed to vary exponentially through the thickness of FG layer. Comparison results were presented to demonstrate the stability and accuracy of the current exact solution. The effects of foundation stiffness, gradient index of material properties of FG layer and coating thickness-to-total thickness ratio on the natural frequencies were considered and discussed. It was observed that the natural frequencies with the softer surface subjected to the foundation differ significantly from that of the plate with the harder surface subjected to the same foundation.

APPENDIX

$$A_{10} = -\mu_0 \qquad A_{20} = \mu_0\left(\alpha_m^2 + \beta_n^2\right) + \frac{\mu_0}{1-2\nu}\left(\alpha_m^2 + \beta_n^2\right) - \rho_0\omega^2 \qquad A_{30} = \frac{\mu_0}{1-2\nu}$$

$$B_{10} = \frac{2-2\nu}{1-2\nu}\mu_0 \qquad B_{20} = -\mu_0\left(\alpha_m^2 + \beta_n^2\right) + \rho_0\omega^2 \qquad B_{30} = \frac{\mu_0}{1-2\nu}\left(\alpha_m^2 + \beta_n^2\right)$$

$$A_{11} = -\mu_0 \qquad A_{21} = -\mu_0\phi \qquad A_{31} = \mu_0\left(\alpha_m^2 + \beta_n^2\right) + \frac{\mu_0}{1-2\nu}\left(\alpha_m^2 + \beta_n^2\right) - \rho_0\omega^2$$

$$A_{41} = \frac{\mu_0}{1-2\nu} \qquad A_{51} = \mu_0\phi$$

$$B_{11} = \frac{2-2\nu}{1-2\nu}\mu_0 \qquad B_{21} = 2\phi\mu_0\left(\frac{1-\nu}{1-2\nu}\right) \qquad B_{31} = -\mu_0\left(\alpha_m^2 + \beta_n^2\right) + \rho_0\omega^2$$

$$B_{41} = \frac{\mu_0}{1-2\nu}\left(\alpha_m^2 + \beta_n^2\right) \qquad B_{51} = \frac{2\nu\phi\mu_0}{1-2\nu}\left(\alpha_m^2 + \beta_n^2\right) \tag{A.1}$$

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