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Dynamic Response of an Axially Moving Viscoelastic Timoshenko Beam

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ABSTRACT

In this paper, the dynamic response of an axially moving viscoelastic beam with simple supports is calculated analytically based on Timoshenko theory. The beam material property is separated to shear and bulk effects. It is assumed that the beam is incompressible in bulk and viscoelastic in shear, which obeys the standard linear model with the material time derivative. The axial speed is characterized by a simple harmonic variation about a constant mean speed. The method of multiple scales with the solvability condition is applied to dimensionless form of governing equations in modal analysis and principal parametric resonance. By a parametric study, the effects of velocity, geometry and viscoelastic parameters are investigated on the response.

Keywords : Viscoelastic; Axially moving beam; Perturbation; Dynamic response; Timoshenko theory.

1 INTRODUCTION

AXIALLY moving beams are used in many engineering devices, such as band saws, aerial cableways, power transmission chains and serpentine belts. Transverse vibrations of these devices are investigated to avoid possible resulting fatigue, failure and low quality.

Chen et al. [1] applied the averaging method to a discretized system via the Galerkin method to present the stability boundaries of axially accelerating viscoelastic beams. Mockensturm and Guo [2] convincingly argued that the axially moving beam should contain the material time derivative to account for the energy dissipation in steady motion. Tang et al. [3] determined transverse nonlinear response of an axially moving Timoshenko elastic beam to external excitations via the method of multiple scales. By combination of the governing equations, they found a single equation and they used the orthogonlity of mode shapes for the solvability condition. Chen et al. [4] investigated the dynamic stability of an axially accelerating viscoelastic beam undergoing parametric resonance by Timoshenko theory. The Kelvin model was used as the constitutive relation for normal and shear stresses with material time derivative. As the solvability condition for two coupled equations, they used the orthogonality of each equations just related to the transverse mode shape. Ding and Chen [5] investigated the steady-state response for an axially moving viscoelastic beam. The method of multiple scales and differential quadrature schemes were applied to the governing equations to investigate the primary resonances under general boundary conditions. They used the Kelvin constitutive model for normal stress-strain relation and Euler-Bernolli (E-B) beam theory. Also, they used the orthogonality property as the solvability condition. Chen et al. [6], investigated the nonlinear parametric

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vibration for axially accelerating viscoelastic beams subjected to parametric excitation. The method of multiple scales and the differential quadrature were employed to analyze the combination and the principal parametric resonances with the focus on steady-state responses. They used the Kelvin model for normal stress-strain relation and orthogonality property for solvability condition. Gayesh [7] investigated the forced dynamics of an axially moving viscoelastic beam with the Kelvin model and employing time derivative in the viscoelastic constitutive relation. The dimensionless partial differential equation of motion is discretized using Galerkin's scheme. The resulting set of equations was solved numerically. Wang et al. [8] investigated the forced vibration of an axially moving viscoelastic beam. They describe the constitutive equation with the standard linear model for normal stress-strain by considering material time derivative. Also they used the E-B beam theory for formulation, multiple scales method to determine the steady-state response and the orthogonality property as the solvability condition. Ghayesh et al. [9] studied the nonlinear coupled longitudinal-transverse vibrations and stability of an axially moving elastic beam, subjected to a harmonic force, which was supported by an intermediate spring, numerically. The equations of motion was discretized using Galerkin's method and the frequency-response curves of the system and the bifurcation diagrams of Poincaré maps were analyzed. Ghayesh et al. [10] examined the nonlinear dynamics of an axially moving viscoelastic beam, while both longitudinal and transverse displacements were taken into account, with employing a numerical technique. The Kelvin model which considers the material time derivative was used in the viscoelastic constitutive relations. The equations of motion for both longitudinal and transverse motions are discretized via Galerkin's method and the resulting equations were solved numerically. Youqi [11] studied the nonlinear parametric vibrations for axially accelerating viscoelastic Timoshenko beams subjected to varying tensions and axial accelerations. He used Timoshenko beam theory and the Kelvin viscoelastic constitutive relation. The governing equation was solved by employing the multiple scales method to investigate the parametric resonances with focus on the steady-state responses.

For analyzing a viscoelastic structure, it is convenient to separate the shear (deviatoric) effects from the purely dilatational (bulk) components. This is due to the fact that in viscoelastic materials, the response to shear can be different from that in bulk. In other words, different types of stress can produce different responses [12]. There is three assumptions for viscoelastic analysis:

- The material behaves viscoelastic in shear and incompressible in dilatation(bulk).
- The material behaves viscoelastic in shear and elastic in bulk.
- The shear and bulk moduli are synchronous.

Each of the common assumptions defines a particular value for either the bulk modulus or Poisson's ratio [13]. So, Kelvin or the other models of viscoelasticity, define the constitutive equation for shear stress- strain and not normal stress- strain. It seems that the most reviewed papers, did not consider this subject.Also, when there is just,a single governing equation, the orthogonality can use as the solvability condition easily. The most paper converted the Timoshenko equations to a single equation and then use the orthogonality condition. This combination is not possible in some cases e.g. when one uses the shear deformation theory. When there is more than one governing equation, it is convenient to use the adjoint functions for the solvability condition.

Determination of dynamic response of an axially accelerated viscoelastic Timoshenko beam with the standard linear model for modal and principal parametric resonance cases is the purpose of this article. We separated the effects of shear and bulk behaviors. We assumed the beam is viscoelastic in shear and incompressible in bulk. The governing equations are coupled and the adjoint functions are used as the solvability conditions. In addition, by the sensitivity analysis, the effects of the geometric and viscoelastic parameters on the response are investigated. We use the perturbation technique up to order-one for analysis. Also we tried to define the perturbation parameter ε as a physical quantity and no just as a bookkeeping value.

2 GOVERNING EQUATIONS

A uniform axially moving beam travels between two supports separated by distance l at the transport time-dependent speed Γ(T). The density is ρ , A= cross section area, I =area moment of inertia and P =axial tension. When the effects of rotary inertia and shear deformation are considered, the bending vibrations can describe by transverse displacement of the mid-plane V(X,T) and its slope φ(X,T). T is time and *X* represents the location of each point. Applying the Newton's second law in the transverse direction and the angular momentum principle yield:

$$\rho A\ddot{V} = PV_{,XX} - Q_{,X} \tag{1a}$$

$$\rho I\,\phi_{,TT} = M_{,X} - Q \tag{1b}$$

By using the material time derivative formula, Eqs. (1) result:

$$\rho A\left(V_{,TT} + 2\Gamma V_{,XT} + \dot{\Gamma} V_{,X} + \Gamma^2 V_{,XX}\right) = PV_{,XX} - Q_{,X} \tag{2a}$$

$$\rho I\left(\phi_{,TT} + k_9(2\Gamma\phi_{,XT} + \dot{\Gamma}\phi_{,X} + \Gamma^2\phi_{,XX})\right) = M_{,X} - Q \tag{2b}$$

where M(X,T) =bending moment, Q(X,T) =shear force and they define as the following [14]:

$$M = \iint \sigma z\, dA = EI\frac{\partial\phi}{\partial X}\ ,\qquad Q = KAG\left(\phi - V_{,X}\right) \tag{3}$$

E =elastic modulus, σ =normal stress, K =shear correction factor and z =distance from the mid-plane. Chen et al. [4] considered Eq. (1b) instead of (2b) for extracting the governing equations. In the other word, they did not consider the material time derivative for Eq. (1b). We will investigate the validity of this assumption later. For this purpose we inserted parameter k_9 which has the values zero or one, in Eq. (2b), We called it “material time derivative coefficient for rotation” in this text. If k_9=1, then the material time derivative for rotation function i.e. φ(X,T) is inserted in formulation and if k_9=0, it is not considered. Also by setting k_9=0, one can compare some of the equations with Chen et al. [4]. For a beam which is viscoelastic in shear and incompressible in bulk the stress-strain relation is [15]:

$$k = \infty, \nu = 0.5 \Rightarrow G = \left.\frac{E}{2(1+\nu)}\right|_{\nu=0.5} = \frac{E}{3} \Rightarrow E = 3G\ ;\ \left\{\begin{matrix} for\ \ viscoelastic\ \ case: p^E\tau = q^E\varepsilon \\ for\ \ elastic\ \ case: \tau = 2G\varepsilon \end{matrix}\right\} \rightarrow G = \frac{q^E}{2p^E} \tag{4}$$

where q^E , p^E =viscoelastic operators and G,k =shear and bulk modulus. So Eqs. (3) lead to the following form:

$$M = (3q^E/2p^E)I\phi_{,X}\quad ,\qquad Q = KA(q^E/2p^E)(\phi - V_{,X}) \tag{5}$$

For a viscoelastic material which obeys the standard linear model, by considering the material time derivative, the viscoelastic operators are:

$$p^E = \eta_1\frac{\partial}{\partial T} + \eta_1\Gamma\frac{\partial}{\partial X} + E_1\,,\quad q^E = \eta_1 E_0\frac{\partial}{\partial T} + \eta_1 E_0\Gamma\frac{\partial}{\partial X} + E_3 \tag{6}$$

where $E_0=E_1+E_2$, $E_3= E_1 E_2$. Fig 1. shows this model.

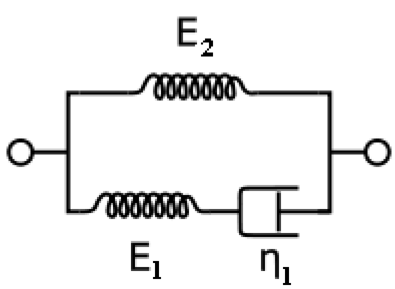

Fig.1
Standard linear model.

By substituting Eqs. (5) into Eqs. (2) the governing equations of an axially moving viscoelastic Timoshenko beam can extract.

3 PERTURBATION METHOD

The perturbation technique is used for analysis. At first, the following parameters are introduced:

$$t^* = \frac{T}{t_0},\ x^* = \frac{X}{l}, V^* = \frac{V}{h_0},\ \varepsilon = \frac{h_0}{l},\ t_0 = \frac{h_0}{\Gamma_0},\ \tau_0 = \frac{\eta_1}{E_1},\ e = \frac{\rho}{E_2}\left(\frac{h_0}{t_0}\right)^2,\ \gamma^* = \frac{\Gamma}{\Gamma_0},$$
$$\tau^* = \frac{\tau_0}{\varepsilon t_0},\ P^* = \frac{P}{AE_2}, G^* = \frac{E_1}{E_2}+1,\ r^* = \frac{I}{Ah_0^2}, \eta = \frac{x^*}{\varepsilon},\ \omega^* = \omega t_0 \tag{7}$$

where t_0 =characteristic time, Γ_0=characteristic velocity (mean axial speed), h_0 =thickness, τ_0 =relaxation time, $(\ )^*$ stands for a dimensionless parameter, e =a dimensionless quantity corresponding to the wave velocity and ε =a small parameter (the ratio of the thickness to the length) which is considered as the perturbation parameter. In the most presented works, ε is a bookkeeping parameter but in this work, it has physical meaning. By using Eqs. (7), the dimensionless form of governing are obtained as the following:

$$\begin{aligned}
&2e\tau^*\varepsilon V^*_{,t^*t^*t^*} + 2eV^*_{,t^*t^*} + 6e\tau^*\gamma^*\varepsilon V^*_{,\eta t^*t^*} + \left(6e\gamma^{*2}\varepsilon - 2P^*\varepsilon - KG^*\varepsilon\right)\tau^* V^*_{,\eta\eta t^*} \\
&+\left(2e\gamma^{*3}\varepsilon - 2P^*\gamma^*\varepsilon - KG^*\gamma^*\varepsilon\right)\tau^* V^*_{,\eta\eta\eta} + \left(-2P^* + 2e\gamma^{*2} - K\right)V^*_{,\eta\eta} \\
&+2e\left(\tau^*\gamma^*\varepsilon\gamma^*_{,t^*t^*} + \gamma^*_{,t^*}\right)V^*_{,\eta} + 6e\tau^*\gamma^*\varepsilon\gamma^*_{,t^*}V^*_{,\eta\eta} + 6e\tau^*\varepsilon\gamma^*_{,t^*}V^*_{,\eta t^*} \\
&+4e\gamma^* V^*_{,\eta t^*} + K\tau^*\gamma^* G^*\varepsilon\phi_{,\eta\eta} + K\phi_{,\eta} + K\tau^* G^*\varepsilon\phi_{,\eta t^*} = 0
\end{aligned} \tag{8a}$$

$$\begin{aligned}
&2e\,\varepsilon\tau^* r^*\phi_{,t^*t^*t^*} + 2er^*\phi_{,t^*t^*} + (2+4k_9)e\varepsilon\tau^* r^*\gamma^*\phi_{,\eta t^*t^*} - 3\varepsilon\tau^* r^*\left(G^* - 2k_9 e\gamma^{*2}\right)\phi_{,\eta\eta t^*} \\
&+\varepsilon\tau^* KG^*\phi_{,t^*} + r^*\left(2k_9 e\gamma^*(3\varepsilon\tau^*\gamma^*_{,t^*} + \gamma^*) - 3\right)\phi_{,\eta\eta} + \varepsilon e\tau^* r^*\gamma^*\left(2k_9\gamma^{*2} - 3G^*\right)\phi_{,\eta\eta\eta} \\
&+K\phi + \left(2k_9 er^*(\varepsilon\tau^*\gamma^*_{,t^*t^*} + \gamma^*_{,t^*}) + \varepsilon\tau^*\gamma^* G^* K\right)\phi_{,\eta} - KV^*_{,\eta} - \tau^*\gamma^*\varepsilon KG^* V^*_{,\eta\eta} \\
&-\tau^* G^* K\varepsilon V^*_{,\eta t^*} + 2k_9 er^*\left(3\varepsilon\tau^*\gamma^*_{,t^*} + 2\gamma^*\right)\phi_{,\eta t^*} = 0
\end{aligned} \tag{8b}$$

Eqs. (8) are coupled differential equations with time dependent coefficients.The axial speed is considered a small simple harmonic variations about a constant mean speed:

$$\gamma^* = 1 + \varepsilon\frac{\gamma_1}{\Gamma_0}\sin\left(\omega^* t^*\right) \tag{9}$$

where $\varepsilon\gamma_1/\Gamma_0$ and ω^* are amplitude and frequency of the axial speed fluctuations, respectively. By defining $T_0 = t^*$ and $T_1 = \varepsilon t^*$, Eqs. (8) convert to multiple-scale form. So, V^* and φ are functions of T_0, T_1, η. **T**he deflection and rotation of the beam is assumed as :

$$\begin{aligned}
&V^*(\eta,t^*;\varepsilon) = v_0(\eta,T_0,T) + \varepsilon v_1(\eta,T_0,T) + O(\varepsilon^2) \\
&\phi^*(\eta,t^*;\varepsilon) = \phi_0(\eta,T_0,T_1) + \varepsilon\phi_1(\eta,T_0,T_1) + O(\varepsilon^2)
\end{aligned} \tag{10}$$

By substituting Eqs. (10) into Eqs. (8) and considering the terms with zero and one orders of ε, we have:
Order-zero equations:

$$Eq1: 2e\frac{\partial^2 v_0}{\partial T_0{}^2}+4e\frac{\partial^2 v_0}{\partial T_0 \partial \eta}+K\frac{\partial \phi_0}{\partial \eta}+\left(-2P^*+2e-K\right)\frac{\partial^2 v_0}{\partial \eta^2}=0,$$
$$Eq2: 2er^*\frac{\partial^2 \phi_0}{\partial T_0{}^2}+(2k_9 e-3)r^*\frac{\partial^2 \phi_0}{\partial \eta^2}+K\phi_0-K\frac{\partial v_0}{\partial \eta}+4k_9 er^*\frac{\partial^2 \phi_0}{\partial \eta \partial T_0}=0 \tag{11}$$

Order-one equations

$$Eq3: 2e\frac{\partial^2 v_1}{\partial T_0{}^2}+4e\frac{\partial^2 v_1}{\partial T_0 \partial \eta}+K\frac{\partial \phi_1}{\partial \eta}+\left(-2P^*+2e-K\right)\frac{\partial v_1}{\partial \eta^2}=\left(-6e+KG^*+2P^*\right)\tau^*\frac{\partial^3 v_0}{\partial T_0 \partial \eta^2}$$
$$-6e\tau^*\frac{\partial^3 v_0}{\partial T_0{}^2 \partial \eta}-4e\frac{\partial^2 v_0}{\partial T_1 \partial \eta}-\frac{2e\gamma_1}{\Gamma_0}\left(2\sin(\omega^* T_0)(\frac{\partial^2 v_0}{\partial T_0 \partial \eta}+\frac{\partial^2 v_0}{\partial \eta^2})+\cos(\omega^* T_0)\omega^*\frac{\partial v_0}{\partial \eta}\right) \tag{12a}$$
$$+\left(-2e+KG^*+2P^*\right)\tau^*\frac{\partial^3 v_0}{\partial \eta^3}-4e\frac{\partial^2 v_0}{\partial T_0 \partial T_1}-KG^*\tau^*\frac{\partial^2 \phi_0}{\partial T_0 \partial \eta}-KG^*\tau^*\frac{\partial^2 \phi_0}{\partial \eta^2}-2e\tau^*\frac{\partial^3 v_0}{\partial T_0{}^3}$$

$$Eq4: 2er^*\frac{\partial^2 \phi_1}{\partial T_0{}^2}+(2k_9 e-3)r^*\frac{\partial^2 \phi_1}{\partial \eta^2}+K\phi_1+4k_9 er^*\frac{\partial^2 \phi_1}{\partial \eta \partial T_0}-K\frac{\partial v_1}{\partial \eta}=-2er^*\tau^*\frac{\partial^3 \phi_0}{\partial T_0{}^3}$$
$$(3G^*-2k_9 e)\tau^* r^*\frac{\partial^3 \phi_0}{\partial \eta^3}-(KG^*\tau^*+2k_9 er^*\frac{\gamma_1}{\Gamma_0}\omega^*\cos(\omega^* T_0))\frac{\partial \phi_0}{\partial \eta}$$
$$-4k_9 er^*\left(\frac{\gamma_1}{\Gamma_0}\sin(\omega^* T_0)(\frac{\partial^2 \phi_0}{\partial \eta \partial T_0}+\frac{\partial^2 \phi_0}{\partial \eta^2})+\frac{\partial^2 \phi_0}{\partial \eta \partial T_1}-\frac{\partial^2 \phi_0}{\partial T_1 \partial T_0}\right)-KG^*\tau^*\frac{\partial \phi_0}{\partial T_0} \tag{12b}$$
$$-(2+4k_9)er^*\tau^*\frac{\partial^3 \phi_0}{\partial \eta \partial T_0{}^2}-3\tau^* r^*(2k_9 e-G^*)\frac{\partial^3 \phi_0}{\partial \eta^2 \partial T_0}+KG^*\tau^*\frac{\partial^2 v_0}{\partial \eta^2}+KG^*\tau^*\frac{\partial^2 v_0}{\partial \eta \partial T_0}$$

Eqs. (11-12) are systems of partial differential equations with constant coefficients. We solve these equations analytically.

4 MODAL ANALYSIS

The response of the beam for $\omega^*=\omega_n$ is considered as an uniform series of the parameter ε as Eqs. (10) where ω_n =a natural frequency. Then the zero and first orders of equations are determined.

4.1 Order ε^0

The solutions of Eq.(11) can assume as:

$$v_0(\eta,T_0,T_1)=\sum_{n=1}^{\infty} v_{00}(\eta,T_1,n)\exp(i\,\omega_n T_0),\ \phi_0(\eta,T_0,T_1)=\sum_{n=1}^{\infty}\phi_{00}(\eta,T_1,n)\exp(i\,\omega_n T_0) \tag{13}$$

For mode *'n'* the n^{th} term of the series is dominant, so just n^{th} term is considered as the response of the beam (mode shape). By substituting Eq. (13) into Eq. (11), a system of ordinary differential equations with constant coefficients (in terms of v_{00} and φ_{00}) is obtained. The solution of this system is considered as:

$$\begin{pmatrix} v_{00} \\ \phi_{00} \end{pmatrix} = V_1 \exp(\beta\eta),\ V_1 = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{14}$$

After substituting, we have:

$$\begin{aligned} &(-2e\,\omega_n{}^2 + 4ei\,\beta\omega_n + (-2P^* + 2e - K)\beta^2)A_1 + K\,\beta A_2 = 0 \\ &-K\,\beta A_1 + (K - 2er^*\omega_n{}^2 + 4k_9\,er^*i\,\beta\omega_n + (2k_9\,e - 3)r^*\beta^2)A_2 = 0 \end{aligned} \tag{15}$$

Nontrival solution of Eqs. (15) corresponds to set the characteristic equation to zero which result $\beta_1..\beta_4$ in terms of ω_n. Also the value of eigenvector V_1 can determine. The response of the system is:

$$\begin{Bmatrix} v_{00}(\eta,T_1,n) \\ \phi_{00}(\eta,T_1,n) \end{Bmatrix} = C_n V_{1|\beta=\beta_1} \exp(\beta_1\eta) + C_2 V_{1|\beta=\beta_2} \exp(\beta_2\eta) + C_3 V_{1|\beta=\beta_3} \exp(\beta_3\eta) + C_4 V_{1|\beta=\beta_4} \exp(\beta_4\eta) \tag{16}$$

C_n, C_2, C_3, C_4 are functions of T_1. By applying the boundary conditions, a system of algebraic equations in the form of $[axx]_{4*4}\ \{c\}=\{0\}_{4*1}$ is created. For nontrival solution, the determinant of [axx] matrix must be vanished. It is a complex algebraic equation of ω_n which can solve with the numerical method. After calculation the eigenvalues, the values C_2, C_3, C_4 are computed in terms of $C_n(T_1)$ (which we dropped index "*n*" for simplicity) and we have:

$$v_{00}(\eta,T_1,n) = C(T_1)v_{0n}(\eta),\quad \varphi_{00}(\eta,T_1,n) = C(T_1)\varphi_{0n}(\eta) \tag{17}$$

4.2 Order ε^1

The solution of Eqs. (12) is assumed as:

$$v_1(\eta,T_0,T_1) = \sum_{n=1}^{\infty} v_{11}(\eta,T_1,n)\exp(i\,\omega_n T_0)\ ,\ \phi_1(\eta,T_0,T_1) = \sum_{n=1}^{\infty} \phi_{11}(\eta,T_1,n)\exp(i\,\omega_n T_0) \tag{18}$$

$C(T_1)$ appears in the non-homogeneous part of Eqs. (12). The solvability condition is used for determining $C(T_1)$. Two adjoint functions $\psi_1(\eta)$, $\psi_2(\eta)$ are multiplied to order-zero equations (Eqs. 11) and integrated from the sum of them over the total domain [16].

$$\int_0^{1/\varepsilon} (\psi_1 \times Eq_1 + \psi_2 \times Eq_2)\,d\eta = 0 \tag{19}$$

In general, by considering Eq. (11,13), if we define Eq_1 and Eq_2 as the following:

$$\begin{aligned} &Eq1: a_2\frac{\partial^2 v_{00}}{\partial\eta^2} + a_1\frac{\partial v_{00}}{\partial\eta} + a_0 v_{00} + b_2\frac{\partial^2\phi_{00}}{\partial\eta^2} + b_1\frac{\partial\phi_{00}}{\partial\eta} + b_0\phi_{00} = 0 \\ &Eq2: c_2\frac{\partial^2 v_{00}}{\partial\eta^2} + c_1\frac{\partial v_{00}}{\partial\eta} + c_0 v_{00} + d_2\frac{\partial^2\phi_{00}}{\partial\eta^2} + d_1\frac{\partial\phi_{00}}{\partial\eta} + d_0\phi_{00} = 0 \end{aligned} \tag{20a}$$

where a_1, a_2, a_3, b_1, b_2 are as the following:

$$a_2 = (-2P^* + 2e - K),\ a_1 = 4ei\,\omega_n, a_0 = -2e\,\omega_n^{\ 2}, b_2 = 0,\ b_1 = K,\ b_0 = 0$$
$$c_2 = 0, c_1 = -K, c_0 = 0, d_2 = (2k_9 e - 3)r^*,\ d_1 = 4k_9 er^* i\,\omega_n, d_0 = K - 2er^*\omega_n^{\ 2} \tag{20b}$$

The adjoint functions ψ_1, ψ_2, are established for all values of v_{00}, φ_{00} as the following:

$$a_2 \frac{\partial^2 \psi_1}{\partial \eta^2} - a_1 \frac{\partial \psi_1}{\partial \eta} + a_0 \psi_1 + c_2 \frac{\partial^2 \psi_2}{\partial \eta^2} - c_1 \frac{\partial \psi_2}{\partial \eta} + c_0 \psi_2 = 0$$
$$b_2 \frac{\partial^2 \psi_1}{\partial \eta^2} - b_1 \frac{\partial \psi_1}{\partial \eta} + b_0 \psi_1 + d_2 \frac{\partial^2 \psi_2}{\partial \eta^2} - d_1 \frac{\partial \psi_2}{\partial \eta} + d_0 \psi_2 = 0 \tag{21a}$$

For a simply supported beam the boundary conditions are $V=0$ and $M = 0 \rightarrow d\phi / d\eta = 0$ (From Eq. (5)) at two ends and the appropriate boundary conditions for $\psi_1(\eta), \psi_2(\eta)$ are set as the following:

$$\left(b_1\psi_1 + d_1\psi_2\right)\Big|_{0,1/\varepsilon} = 0, \quad \frac{\partial \psi_2}{\partial \eta}\Big|_{0,1/\varepsilon} = 0 \tag{21b}$$

The solution of Eq. (21a) is considered as:

$$\psi = V_2 \exp(m\,\eta),\ V_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} \tag{22}$$

So, we have:

$$[D_2]\{\psi\}'' + [D_1]\{\psi\}' + [D_0]\{\psi\} = \{0\}_{2*1},\ \{\psi\} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix},$$
$$[D_2] = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, [D_1] = \begin{bmatrix} -a_1 & -b_1 \\ -c_1 & -d_1 \end{bmatrix}, [D_0] = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \tag{23}$$

The eigenvector is :

$$V_2 = \begin{pmatrix} b_2 m^2 + b_1 m + b_0 \\ -(a_2 m^2 + a_1 m + a_0) \end{pmatrix} v_{22} \tag{24}$$

The solution of Eq. (23) is as the following:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = d_1 V_{2|m=m_1} \exp(m_1\eta) + d_2 V_{2|m=m_2} \exp(m_2\eta) + d_3 V_{2|m=m_3} \exp(m_3\eta) + d_4 V_{2|m=m_4} \exp(m_4\eta) \tag{25}$$

By applying the boundary condition Eqs. (21b) the coefficients d_2, d_3, d_4 are obtained in terms of d_1. After determination the adjoint functions, these functions are multiplied into Eqs. (12) and then integrated from the sum of them.

$$\int_0^{1/\varepsilon} \left(\psi_1 \times Eq_3 + \psi_2 \times Eq_4\right) d\eta = \int_0^{1/\varepsilon} \left(\psi_1 \times f_1 + \psi_2 \times f_2\right) d\eta \tag{26a}$$

Eq_3, Eq_4 are the left hand side of the first order Eqs. (12). From Eqs. (21) it results that the left hand side of Eq. (26a) is zero. f_1, f_2 are the coefficients of $\exp(\pm i\omega_n T_0)$ terms which can produce secular terms. We have:

$$f_1 : b_{11} C(T_1) + b_{33}\frac{dC(T_1)}{dT_1}, \quad f_2 : b_{22} C(T_1) + b_{44}\frac{dC(T_1)}{dT_1},$$

$$b_{11} = e_1\frac{\partial^3 v_0}{\partial\eta^3} + i\,\omega_n e_2\frac{\partial^2 v_0}{\partial\eta^2} + e_3\frac{\partial v_0}{\partial\eta} + e_4 v_0 + e_5(\frac{\partial^2\phi_0}{\partial\eta^2} + i\,\omega_n\frac{\partial\phi_0}{\partial\eta}),\ b_{33} = -4e(\frac{\partial v_0}{\partial\eta} + i\,\omega_n v_0) \tag{26b}$$

$$b_{22} = s_1\frac{\partial^3\phi_0}{\partial\eta^3} + s_2\frac{\partial^2\phi_0}{\partial\eta^2} + s_3\frac{\partial\phi_0}{\partial\eta} + s_4\phi_0 - e_5(\frac{\partial^2 v_0}{\partial\eta^2} + i\,\omega_n\frac{\partial v_0}{\partial\eta}),\ b_{44} = -4er^*i\,\omega_n\phi_0 - 4k_9 er^*\frac{\partial\phi_0}{\partial\eta}$$

where:

$$\begin{aligned} &e_1 = \tau^*(-2e + KG^* + 2P^*),\ e_2 = \tau^*(-6e + KG^* + 2P^*),\ e_3 = 6e\tau^*\omega_n^{\,2}, \\ &e_4 = 2e\tau^* i\,\omega_n^{\,3}, e_5 = -KG^*\tau^*, s_1 = \tau^* r^*(3G^* - 2k_9 e),\ s_2 = 3i\,\omega_n\tau^* r^*(G^* - 2k_9 e), \\ &s_3 = \tau^*((2+4k_9)r^* e\omega_n^{\,2} - KG^*), s_4 = -\tau^* i\,\omega_n(KG^* - 2r^* e\omega_n^{\,2}) \end{aligned} \tag{26c}$$

The right hand side of Eq. (26a) is :

$$\int_0^{1/\varepsilon}(\psi_1\times f_1 + \psi_2\times f_2)d\eta = \int_0^{1/\varepsilon}\left((\psi_1 b_{11} + \psi_2 b_{22})C(T_1) + (\psi_1 b_{33} + \psi_2 b_{44})\frac{dC(T_1)}{dT_1}\right)d\eta \tag{27}$$

After integrating, we have

$$\frac{dC(T_1)}{dT_1} + \kappa_1 C(T_1) = 0, \kappa_1 = \frac{Q_1}{W_1}, Q_1 = \int_0^{1/\varepsilon}(\psi_1 b_{11} + \psi_2 b_{22})d\eta,\ W_1 = \int_0^{1/\varepsilon}(\psi_1 b_{33} + \psi_2 b_{44})d\eta \tag{28}$$

Eq. (28) is a first order differential equation which can solve for $C(T_1)$.The particular solution is as the following form:

$$\begin{Bmatrix} v_{1p} \\ \phi_{1p} \end{Bmatrix} = \begin{Bmatrix} A1(\eta) \\ B1(\eta) \end{Bmatrix}\exp(2i\,\omega_n T_0) + \begin{Bmatrix} A2(\eta) \\ B2(\eta) \end{Bmatrix} \tag{29}$$

By substituting Eq. (29) into the first order equations (Eqs.12), $A1(\eta)$, $A2(\eta)$, $B1(\eta)$, $B2(\eta)$ can determine. The total response is:

$$\begin{Bmatrix} v_1 \\ \phi_1 \end{Bmatrix} = \begin{Bmatrix} v_g \\ \phi_g \end{Bmatrix}\exp(i\,\omega_n T_0) + \begin{Bmatrix} v_{1p} \\ \phi_{1p} \end{Bmatrix} \tag{30a}$$

v_g and φ_g are homogenous solutions in the form of Eq. (16). The boundary conditions are:

$$v_1 = 0,\ \partial\varphi_1/\partial\eta = 0 \quad at\ \eta = 0, 1/\varepsilon \tag{30b}$$

5 PRINCIPAL PARAMETRIC RESONANCE

If the dimensionless natural frequency ω^* approaches to two-times of natural frequency of generating autonomous linear system Eq. (11) the principal parametric resonance may occur. A detuning parameter μ is introduced to quantify the deviation of ω^* from $2\omega_n$ as the following:

$$\omega^* = 2\omega_n + \varepsilon\mu \tag{31}$$

We investigate the order-zero and one equations (Eqs. (11-12)). The order- zero Eqs. (11) are homogenous and the solution is:

$$\begin{aligned} v_0(\eta,T_0,T_1) &= \sum_{n=1}^{\infty} C_n(T_1) v_{0n}(\eta)\exp(i\,\omega_n T_0) + \overline{C_n(T_1)}\,\overline{v_{0n}(\eta)}\exp(-i\,\omega_n T_0) \\ \phi_0(\eta,T_0,T_1) &= \sum_{n=1}^{\infty} C_n(T_1)\phi_{0n}(\eta)\exp(i\,\omega_n T_0) + \overline{C_n(T_1)}\,\overline{\phi_{0n}(\eta)}\exp(-i\,\omega_n T_0) \end{aligned} \tag{32}$$

where "bar" stands for the complex conjugate and for simplicity, we dropped index "n". The procedure of determining $C(T_1)$ is similar to modal analysis i.e. the adjoint function has to distinguish. We substitute Eqs. (31-32) into Eqs. (12). The coefficients of the secular terms in non-homogenous part of Eqs. (12) are as the following:

$$\begin{aligned} f_{11} &: b_{11}C(T_1) + b_{33}\frac{dC(T_1)}{dT_1} + b_{55}\overline{C(T_1)}e^{i\mu T_1}, b_{55} = 2e\,i\,\frac{\gamma_1}{\gamma_0}\frac{\partial^2 \overline{v_0}}{\partial\eta^2} \\ f_{22} &: b_{22}C(T_1) + b_{44}\frac{dC(T_1)}{dT_1} + b_{66}\overline{C(T_1)}e^{i\mu T_1}, b_{66} = 2k_9\,e\,i\,r^*\frac{\gamma_1}{\gamma_0}\frac{\partial^2 \overline{\phi_0}}{\partial\eta^2} \end{aligned} \tag{33a}$$

f_{11}, f_{12} are the coefficients of $\exp(\pm i\,\omega_n T_0)$ terms. According to Eq. (26a) we have:

$$\int_0^{1/\varepsilon} (\psi_1 \times f_{11} + \psi_2 \times f_{12})\,d\eta = 0 \tag{33b}$$

After integration we have:

$$\frac{dC(T_1)}{dT_1} + \kappa_1 C(T_1) + \chi\overline{C(T_1)}\exp(i\,\mu T_1) = 0,\quad \chi = \frac{Q_2}{W_2}, Q_2 = \int_0^{1/\varepsilon}(\psi_1 b_{55} + \psi_2 b_{66})d\eta, W_2 = \int_0^{1/\varepsilon}(\psi_1 b_{33} + \psi_2 b_{44})d\eta \tag{34}$$

where and $b_{11}..b_{44}$, κ were defined in Eqs. (26). If $C(T_1) = a_n(T_1)\exp(i\,\beta_1(T_1))$ then Eq.(34), results:

$$(\dot{a}_n + a_n i\,\dot{\beta}_1 + \kappa_1 a_n) + \chi\overline{a_n}\exp(i\,\mu T_1)\exp(-2i\,\beta_1) = 0 \tag{35}$$

We assume $\beta_1 = \mu T_1/2$ and Eq.(35) simplifies as:

$$\dot{a}_n + (\kappa_1 + i\,\mu/2)a_n + \chi\overline{a_n} = 0 \tag{36}$$

By considering the real (Re) and imaginary (Im) parts of a_n as $a_n = p(T_1) + i\,q(T_1)$ and assuming zero initial condition $p(0)=0$, a_n is obtained from Eq. (36). So, $C(T_1)$ can determine. After removing the secular terms, the total solution of Eqs. (12), is as the following:

$$\begin{Bmatrix} v_1 \\ \phi_1 \end{Bmatrix} = \begin{Bmatrix} v_g \\ \phi_g \end{Bmatrix} \exp(i\ \omega_n T_0) + \begin{Bmatrix} v_{1p} \\ \phi_{1p} \end{Bmatrix}, \quad \begin{Bmatrix} v_{1p} \\ \phi_{1p} \end{Bmatrix} = \begin{Bmatrix} A1(\eta) \\ B1(\eta) \end{Bmatrix} \exp(3i\ \omega_n T_0) \exp(i\ \mu T_1) \tag{37}$$

v_g and φ_g are homogenous solutions of Eqs. (12).

6 DISCUSSION

By a parametric study, the effects of mechanical and geometrical parameters on the response in modal and principal parametric resonance are investigated. Table 1. reports the beam properties.

Table1
Beam characteristic.

Length	l=1 (m)
Width	b=0.04 (m)
Thickness	h_0=0.004 (m)
Density	ρ=7800 (kg/m^3)
Initial tension force	P=100 (N)
Modulus of viscoelastic model	E_1=1e10, E_2=0.33e10 (Pa)
Viscosity coefficient	η_1=0.25e9 (Pa.s)
Mean speed	Γ_0=0.5 (m/s)
Amplitude of velocity fluctuations	γ_1/Γ_0=0.5
Shear correction factor	K=0.83
material time derivative coefficient	$k_9=1$

6.1 Modal analysis

Fig. 2 shows the mode shape for T_0={0.5,1,3,5}. All the figures related to mode 2.

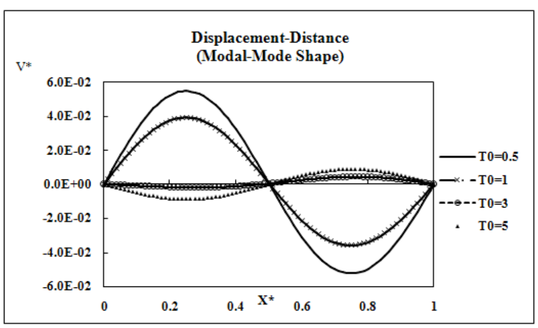

Fig.2
Mode shape in different times for modal analysis (mode 2).

Fig. 3 shows the time response in the middle point of the beam for Kelvin, standard linear and elastic models. The response for the linear standard and Kelvin models are very close but the deflection in elastic case is more than the viscoelastic models. Figs. 4-5 show the effect of axial tension and mean axial speed on the response. Increasing the axial tension decreases the displacement due to increasing the stiffness and increasing the mean axial speed increases it. The axial mean speed does not have significant effect on the response for Γ_0< 0.5 (m/s). Fig. 6 shows the effect of speed amplitude on the response. Increasing the amplitude, decreases the response for small T_0 but for large T_0, The solution is not sensitive to this quantity. Fig. 7 shows the effect of width on the response. By increasing the width, the displacement increases. Fig. 8 shows the effect of thickness on the response. The displacement is very sensitive to the thickness. By increasing the thickness, the decay rate decreases very fast. According to Fig. 9 by increasing the viscosity coefficient, the response value decreases. It corresponds to increasing the relaxation time. In our problems, this quantity affects significantly on the response approximately in the range 1e7< η_1 < 1e10 (Pa.s). For η_1>1e10 there is a heavy damping on the system. From Fig. 10 the response is very sensitive to the elastic modulus E_2. Increasing of this quantity decreases the period of oscillations. The calculations show that the response does not change by E_1 (1e8< E_1 <1e14 Pa). This graph has not shown.

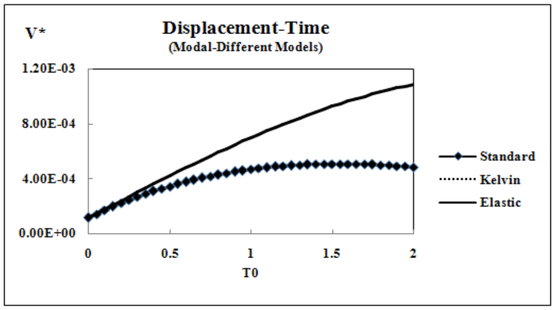

Fig.3
Response of middle point for different models.

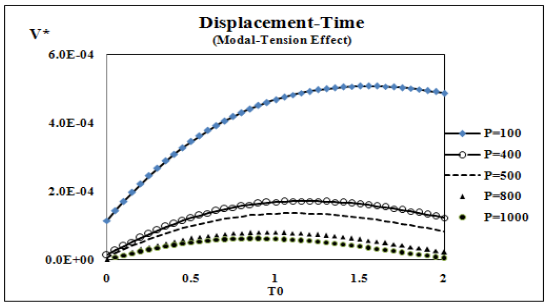

Fig.4
Effect of tension (in terms of Newton) on response.

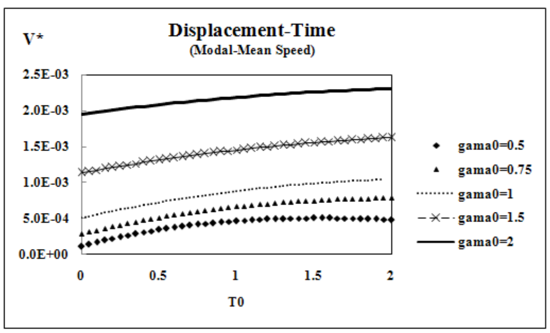

Fig.5
Effect of mean axial speed (m/s) on response.

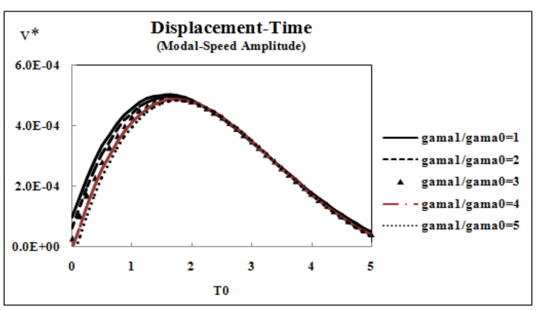

Fig.6
Effect of speed amplitude on response.

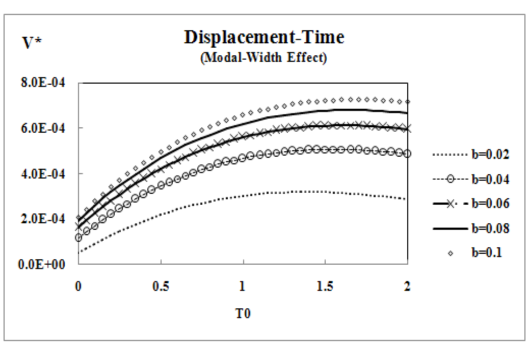

Fig.7
Effect of width (m) on response.

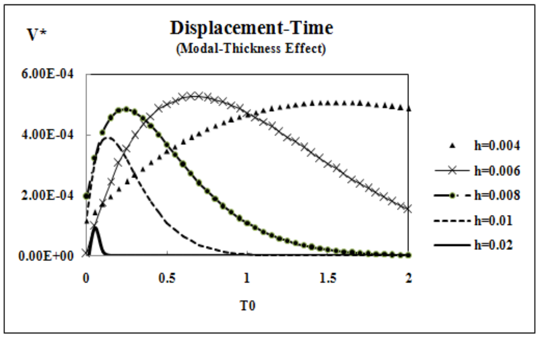

Fig.8
Effect of thickness (m) on response.

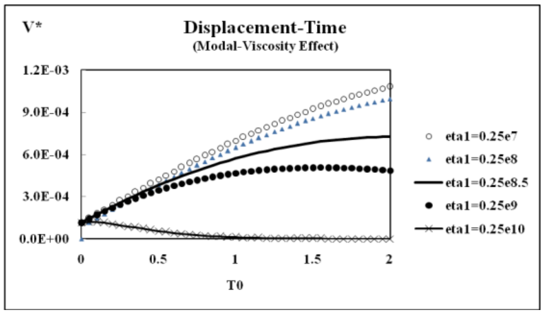

Fig.9
Effect of viscoelastic coefficient (Pa.s) on response.

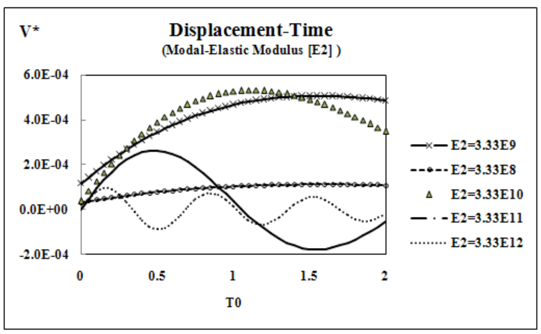

Fig.10
Effect of elastic modulus E_2 (Pa) on response.

The variations of displacement with tension *P*, mean axial speed Γ_0 , width *b*, density ρ for a typical time is approximated with a power trend line as $V^{*}=b\ .\ x^{c}$, where *x* is the mentioned parameters. The values *c* for T_0=0.5,1 has been listed in Table 2. The other parameters for each case, is according to Table 1. The variation of the displacement with the speed amplitude γ_1/Γ_0 is linear with a small slope. For example for $T_0=0.5$ we have $V^{*}= -2e\text{-}5{*}\gamma_1/\Gamma_0 + c_1$ where c_1 is a constant.

Table 2
Curve-fitting parameter *c*

Parameter *(x)*	$c\ (T_0=0.5)$	$c\ (T_0=1)$	Reference
P	-0.82	-0.86	Fig. 4
Γ_0	1.310	1.108	Fig. 5
b	0.515	0.486	Fig. 7

6.2 Principal parametric resonance

Fig. 11 shows the response of middle point of the beam for different models. Similar to the modal analysis the displacement for viscoelastic models is less than the elastic case. Fig. 12 shows the response with respect to the relaxation time. Decreasing the relaxation time corresponds to decreasing the displacement.

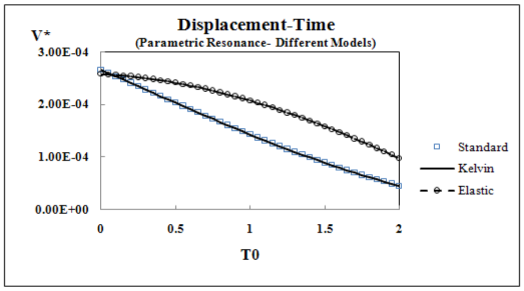

Fig.11
Response of middle point for different models (parametric resonance).

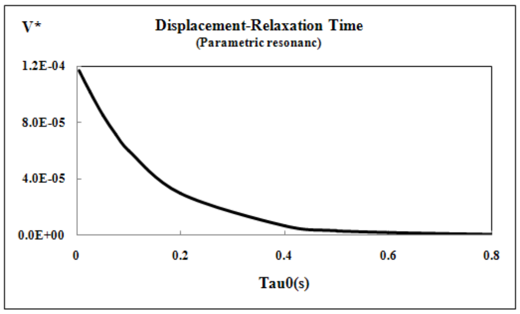

Fig.12
Displacement of middle point in terms of relaxation time (parametric resonance).

The presented results were based on two-term expansion in Eqs. (10). Fig. 13 shows the displacements for one-tem ($V^{*}=v_0$) and two-terms ($V^{*}=v_0+\varepsilon\, v_1$) in Eqs. (10). It is seen that there is a good convergence in response for order-one in Eqs. (10).

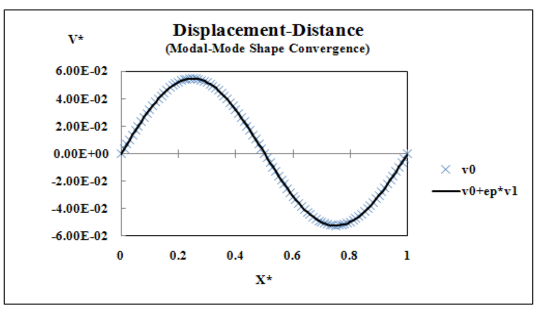

Fig.13
Effect of different orders on displacements.

Table 3
Effect of material time derivative parameter (k_9) on natural frequency.

a	2	3	4	5	6	7	8	9	10	11
k_9=1	0.4075	0.5077	0.5753	0.4485	0.4486	0.4495	0.4580	0.5355	1.0263	2.9524
k_9=0	0.4485	0.4485	0.4485	0.4485	0.4486	0.4495	0.4580	0.5355	1.0263	2.9524

$E_2 = 3.330\times 10^{a}$; $e = 5.856\times 10^{2-a}$

Table 3. reports the effect of material time derivative k_9 on natural frequency. In the derived formulas, ther is the term $e.k_9$. The value of this quantity can affect on determination of frequency. For small values of e, the value of k_9 cannot affect on the calculations. It corresponds to large values of "a" in Table 3. But for small values of "a" , considering the material time derivative parameter is essential. By decreasing "a", the model (Fig. 1) approaches to Maxwell model. So, inserting material time derivative parameter in formulation is very important for small values "a" especially for Maxwell model. In Fig.1 when E_2 approaches to infinity, the model is converted to Kelvin model, so according the presented calculations, the parameter k_9 is not important to calculate the natural frequency. Chen et al. [4] used the Kelvin model and they set k_9=0 or their calculations are acceptable for the used model. Due to depending of the response to natural frequency, one can expect the similar result for the response. The results of Table 3 are based on Table 1 data (except that E_2) and the calculations performed using the presented algorithm by Seddighi and Eipakchi [17].

7 CONCLUSIONS

In this paper the dynamic response of an axially accelerated viscoelastic beam was determined analytically. The beam modeled by considering the Timoshenko beam theory, separating the effects of shear and bulk behavior, using the standard linear model for shear behavior, the material time derivative and the harmonically axial speed fluctuating about a constant mean value. The governing equations are coupled and the adjoint functions are used as the solvability conditions. Then the response of the beam was obtained in both modal and principal parametric resonance by using the multiple-scale method. Finally by a parametric study, the effects of mechanical and geometrical parameters on the response in modal and principal parametric resonance, demonstrate the following conclusions.

- Increase the viscoelastic coefficient decreases the response for special range. The modulus of elasticity E_1 has no effect on the displacement, but the modulus of elasticity E_2 has significant effect on the response.
- Increasing the axial velocity increases the displacement, but amplitude of speed fluctuations has no influence on the displacement for large T_0.
- Increasing the width increases the displacement. The thickness variations has much effect on displacement.
- Increasing the axial tension decreased the displacement.
- The difference in response between Kelvin model and standard linear model is about 0.06% so, the standard model can not improve the result.
- Difference in response between elastic case and standard linear model is about 65.5% in modal analysis and 40% in principal parametric resonance so, by assumption the viscosity for structure, the displacements decrease.
- There is a little differences between order-zero perturbation and first order of it, so considering just two terms in perturbation expansion is enough for convergence.

Inserting the material time derivative for rotation is important especially for small values E_2.

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