

An Investigation into Resonant Frequency of Triangular V-Shaped Cantilever Piezoelectric Vibration Energy Harvester

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ABSTRACT

Power supply is a bottle-neck problem of wireless micro-sensors, especially where the replacement of batteries is impossible or inconvenient. Now piezoelectric material is being used to harvest vibration energy for self-powered sensors. However, the geometry of a piezoelectric cantilever beam will greatly affect its vibration energy harvesting ability. This paper deduces a remarkably precise analytical formula for calculating the fundamental resonant frequency of V-shaped cantilevers using Rayleigh-Ritz method. This analytical formula, which is very convenient for mechanical energy harvester design based on Piezoelectric effect, is then validated by ABAQUS simulation. This formula raises a new perspective that, among all the V-shaped cantilevers and in comparison with rectangular one, the simplest tapered cantilever can lead to maximum resonant frequency and highest sensitivity. © 2016 IAU, Arak Branch. All rights reserved.

Keywords : Mechanical energy harvester; Piezoelectric; V-shaped cantilever; Resonant frequency; Finite element.

1 INTRODUCTION

ENERGY harvesting (also known as energy scavenging) has been around for decades. To feed the world's needs for energy, macro- scale energy harvesting technologies have successfully established. On the other hand, for low powered electronics devices, harvesting energy from the ambient vibrations seems to be an ideal solution due to the definite life span and high cost for replacement of the traditional batteries. Three mechanisms are available for vibration energy harvesting; using electrostatic devices, electromagnetic field and utilizing piezoelectric based materials. The performance of piezoelectric vibration energy harvesters is more often than other methods. Compared to other structural forms of beams, a cantilever beam can obtain the maximum deformation and strain under the same conditions. The larger deflection leads to more stress, strain, and consequently a higher output voltage and power. Therefore the vast majority of piezoelectric vibration energy harvesting devices use a cantilever beam structure. [1-4]. A cantilever-type energy harvester has been intensively studied, and a tapered cantilever has been found to be the optimum design [5], because it ensures a large constant strain (and a large power output) in the piezoelectric layer.

Most of the previous research works focused on designing a linear vibration resonator, in which the maximum system performance can be achieved when the energy harvester is tuned to match its resonance frequency with the

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external excitation frequency. If the excitation frequency slightly shifts, the performance of the harvester will dramatically decrease. Since in the majority of practical cases, the vibration in the environment is frequency-varying or totally random with the energy distributed in a wide spectrum, how to broaden the bandwidth of harvesters becomes one of the most challenging issues before their practical deployment [6].

In practice, the energy harvester is a multi-degree-of-freedom system or a distributed parameter system. Certain vibration mode can be excited when the driving frequency approaches one natural frequency of the harvester. If multiple vibration modes of the harvester are utilized, useful power can be harvested over multiple frequency spectra, that is, wider bandwidth can be covered for efficient energy harvesting. Rather than discrete bandwidth due to the multiple modes of a single beam, multiple cantilevers or cantilever array integrated in one energy harvesting device can provide continuous wide bandwidth, if the geometric parameters of the harvester are appropriately selected [6-8]. Accordingly, by division of a triangular beam into some V-shaped beams, can be found in an array of beams that can cover a wider range of frequencies (Fig.1) [9-11].

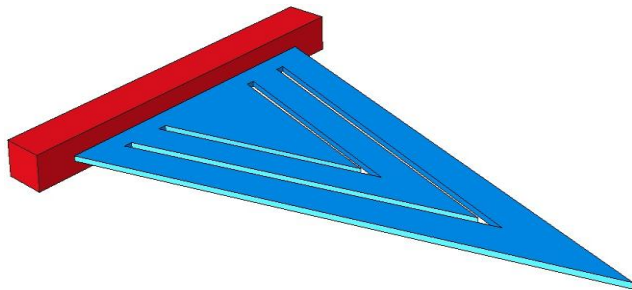


Fig.1
Division of a triangular beam into some V-shaped beams.

The geometry of a piezoelectric cantilever beam will greatly affect its vibration energy harvesting ability. The sensitivity of resonant cantilever piezoelectric energy harvesters is directly proportional to the resonant frequency. So far, the calculation of resonant frequency of V-shaped cantilevers has seldom been reported in the literature [12]. In order to calculate the resonant frequency of V-shaped cantilevers, this paper deduces a highly precise analytical formula using Rayleigh-Ritz method, and then introduces the optimization method for enhancing the resonant frequency with this formula. If a small number of lowest natural frequencies of the system is required, the Rayleigh-Ritz method can be used. The Rayleigh-Ritz method can be considered as an extension of Rayleigh's method and it use a series of assumed functions that satisfies kinematic BC and find coefficients by minimizing Rayleigh quotient. The method is based on the fact that Rayleigh's quotient gives an upper bound for the first eigenvalue. This useful analytical formula, is confirmed by simulation results in ABAQUS 14.1 software, and presents a strong potential to be used in the design and optimization of V-shaped cantilever piezoelectric energy harvesters. It is noteworthy that a cantilever beam can have many different modes of vibration, each with a different resonant frequency. The first mode of vibration has the lowest resonant frequency, and typically provides the most deflection and therefore electrical energy. Accordingly, energy harvesters are generally designed to operate in the first resonant mode.

This research proposes a new design for a cantilever-type piezoelectric energy harvester called V-shaped cantilever and the main focus of this paper is to study the resonant frequency of the new design in piezoelectric mechanical energy harvester.

2 THEORETICAL ANALYSIS

2.1 Deflection function of rectangular cantilevers

Fig.2 shows the structure of single-layered or multi-layered rectangular cantilever with length L , width W , thickness H , effective density ρ and effective Young's modulus E . When applying a normal force F at the free end of the cantilever, the differential equation of the cantilever can be expressed as [13]

$$\frac{d^2z(x)}{dx^2} = \frac{F(L-x)}{EI} = \frac{12F(L-x)}{EWH^3} \quad (1)$$

where x is the distance from the fixed end, and $I = WH^3 / 12$ is the cross-sectional area moment of inertia.

As one end of the cantilever is fixed, the corresponding boundary conditions are;

$$z(0) = 0 \tag{2}$$

and

$$\left. \frac{dz(x)}{dx} \right|_{x=0} = 0 \tag{3}$$

The solution of Eq. (1) - Eq. (3) can be expressed as;

$$z(x) = \frac{2Fx^2(3L-x)}{EWH^3} = Ax^2(3L-x) \tag{4}$$

This is the deflection function along the length direction where A is a constant.

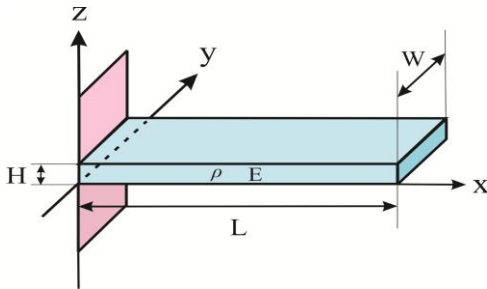


Fig.2
Schematic drawing of a cantilever beam.

2.2 Resonant frequency of cantilevers with arbitrary shapes

When considering the resonant behavior of a cantilever with an arbitrary shape whose width function is $W(x)$, the deflection function of Eq. (4) can be used as the mode shape. The displacement forms in a Rayleigh-Ritz procedure must be continuous and satisfy all geometric constraints. So the vibration displacement at each position can be written as;

$$z(x,t) = Ax^2(3L-x)\sin(\omega t + \alpha) \tag{5}$$

where A and α are constants, t is the time, and $\omega=2\pi f$ is the angular frequency.

The kinetic energy of the system is [14];

$$\begin{aligned} T &= \int_0^L \frac{1}{2} [\rho HW(x) dx] \left(\frac{\partial z}{\partial t} \right)^2 = \int_0^L \frac{1}{2} \rho HW(x) [\omega Ax^2(3L-x)\cos(\omega t + \alpha)]^2 dx \\ &= \frac{1}{2} \rho H \omega^2 A^2 \cos^2(\omega t + \alpha) \int_0^L W(x) x^4 (3L-x)^2 dx \end{aligned} \tag{6}$$

So the maximum kinetic energy of the system is;

$$T_{\max} = \frac{1}{2} \rho H \omega^2 A^2 \int_0^L W(x) x^4 (3L-x)^2 dx \tag{7}$$

The potential energy of the system is [14];

$$V = \int_0^L \frac{1}{2} EI(x) \left(\frac{\partial^2 z}{\partial x^2} \right)^2 dx = \int_0^L \frac{1}{2} E \frac{W(x)H^3}{12} 6A(L-x)\sin^2(\omega t + \alpha) dx = \frac{3}{2} EH^3 A^2 \sin^2(\omega t + \alpha) \int_0^L W(x)(L-x)^2 dx \tag{8}$$

where $I(x) = W(x)H^3 / 12$ is the cross-sectional area moment of inertia. Therefore, the maximum potential energy of the system is;

$$V_{\max} = \frac{3}{2}EH^3A^2 \int_0^L W(x)(L-x)^2 dx \tag{9}$$

According to conservation law of mechanical energy;

$$T_{\max} = V_{\max} \tag{10}$$

Hence, the resonant frequency can be obtained as;

$$f(W(x)) = \frac{\omega}{2\pi} = \frac{H}{2\pi} \sqrt{\frac{3E}{\rho}} \sqrt{\frac{\int_0^{L_1} W(x)(L-x)^2 dx}{\int_0^{L_1} W(x)x^4(3L-x)^2 dx}} \tag{11}$$

In particular, for the case of a rectangular cantilever with length L_1 , width W_1 , thickness H , density ρ and Young's modulus E , the resonant frequency can be deduced from Eq. (11);

$$f_{rect} = \frac{H}{2\pi} \sqrt{\frac{3E}{\rho}} \sqrt{\frac{\int_0^{L_1} W_1(L_1-x)^2 dx}{\int_0^{L_1} W_1x^4(3L_1-x)^2 dx}} = \frac{\sqrt{1155}}{66\pi} \frac{H}{L_1^2} \sqrt{\frac{E}{\rho}} \approx 0.1639 \frac{H}{L_1^2} \sqrt{\frac{E}{\rho}} \tag{12}$$

2.3 Resonant frequency of V-shaped cantilevers

Fig.3(a) shows that a typical V-shaped cantilever can be treated as the difference between two triangular cantilevers, with lengths L_0 and L_1 , and with widths W_0 and W_1 respectively. It can be easily confirmed by Eq. (11), that due to the mirror symmetry of V-shaped cantilever, we need only analyze half of it, which is a quadrilateral cantilever as shown in Fig.3(b).

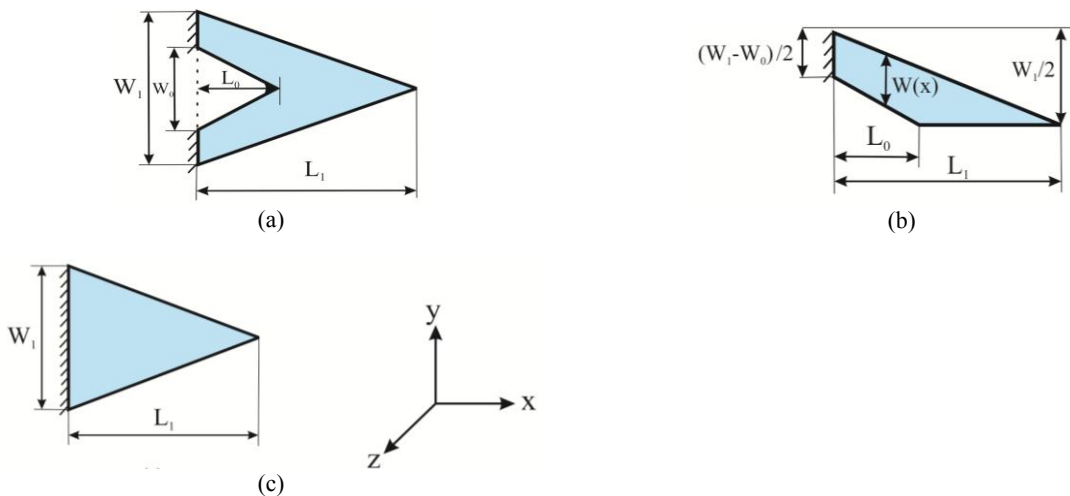


Fig.3 Shape and dimension of (a) V-shaped cantilever (b) Half of the V-shaped cantilever (c) Tapered cantilever.

Obviously, the width function of the quadrilateral cantilever is a piecewise-continuous function of x , that is;

$$W(x) = \begin{cases} \frac{W_1}{2} \left(1 - \frac{x}{L_1}\right) - \frac{W_0}{2} \left(1 - \frac{x}{L_0}\right), & x \in [0, L_0] \\ \frac{W_1}{2} \left(1 - \frac{x}{L_1}\right), & x \in [L_0, L_1] \end{cases} \tag{13}$$

For calculation convenience, it is reasonable to define the width ratio a and the length ratio b of the two tapered cantilevers;

$$a = \frac{W_0}{W_1}, b = \frac{L_0}{L_1} \tag{14}$$

Substituting Eq. (13) and Eq. (14) into Eq. (11), the resonant frequency formula of the quadrilateral cantilever (just the resonant frequency of V-shaped cantilever) is obtained.

$$f(W(x)) = \frac{H}{2\pi} \sqrt{\frac{3E}{\rho}} \sqrt{\frac{\int_0^{L_1} W(x)(L_1-x)^2 dx}{\int_0^{L_1} W(x)x^4(3L_1-x)^2 dx}} = \frac{H}{2\pi} \sqrt{\frac{3E}{\rho}} \sqrt{\frac{\frac{W_1 L_1^3}{8} - \frac{W_0 L_1^2 L_0}{4} + \frac{W_0 L_1 L_0^2}{6} - \frac{W_0 L_0^3}{24}}{\frac{7W_1 L_1^7}{80} - \frac{3W_0 L_1^2 L_0^5}{20} + \frac{W_1 L_1 L_0^6}{14} - \frac{W_0 L_0^7}{112}}} \tag{15}$$

$$= \frac{H}{2\pi} \sqrt{\frac{3E}{\rho}} \sqrt{\frac{70(3W_1 L_1^3 - 6W_0 L_1^2 L_0 + 4W_0 L_1 L_0^2 - W_0 L_0^3)}{3(49W_1 L_1^7 - 84W_0 L_1^2 L_0^5 + 40W_0 L_1 L_0^6 - 5W_0 L_0^7)}} = \frac{H}{2\pi L_1^2} \sqrt{\frac{70E}{\rho}} \sqrt{\frac{3 - 6ab + 4ab^2 - ab^3}{49 - 84ab^5 + 40ab^6 - 5ab^7}}$$

In order to represent the relationship between the resonant frequency and the two ratios a and b , we can define a characteristic function;

$$g(a,b) = \sqrt{\frac{3 - 6ab + 4ab^2 - ab^3}{49 - 84ab^5 + 40ab^6 - 5ab^7}} \quad a \in [0,1], b \in [0,1] \tag{16}$$

Thus, the resonant frequency of V-shaped cantilever is;

$$f(W(x)) = \frac{H}{2\pi L_1^2} \sqrt{\frac{70E}{\rho}} g(a,b) \tag{17}$$

As shown in Fig.4, $g(a,b)$ reaches the maximum value $\frac{\sqrt{3}}{7} \approx 0.2474$, when $b=0$ or $b=1$ or $a=0$. That means V-shaped cantilever achieves maximum resonant frequency only when $L_0=0$ or $L_0=L_1$ or $W_0=0$. Apparently, when $L_0=0$ or $W_0=0$, the V-shaped cantilever turns into a tapered cantilever as shown in Fig.3(c). When $L_0=L_1$, the V-shaped cantilever turns into two side by side tapered cantilevers, however, this peculiar shape is difficult to carry out in practice.

Anyway, tapered cantilever, a special kind of V-shaped cantilever and easy for micro-fabrication, can reach the maximum resonant frequency and thus the highest sensitivity.

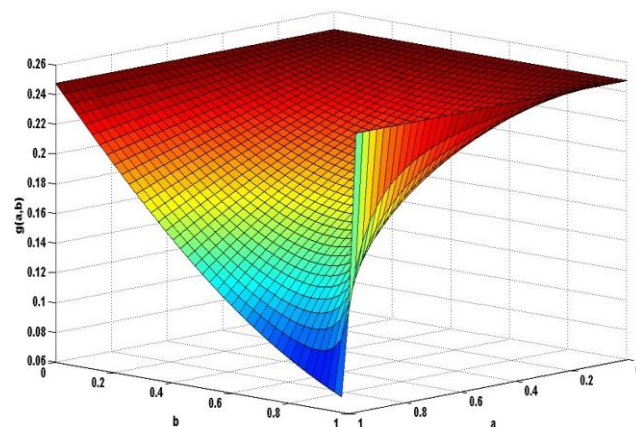


Fig.4
The function image of $g(a,b)$.

3 VERIFICATION BY SIMULATION RESULTS

In order to assess the accuracy of Eq. (17), relative error δ is introduced to compare the calculation results using this formula with the corresponding simulation results.

$$\delta = \frac{f - f'}{f} \quad (18)$$

where f refers to the calculation results with Eq. (17), and f' refers to simulation results with ABAQUS modal analysis. Consider a series of V-shaped cantilevers with different shapes, assuming, $\rho=10 \text{ kg/m}^3$, $E=10^{10} \text{ Pa}$, $H=1 \text{ mm}$, $W_1=80 \text{ mm}$, $W_0=40 \text{ mm}$, $L_1=100 \text{ mm}$ and changing L_0 , the calculation according to Eq. (17) and the corresponding simulation results with ABAQUS are listed in Table 1.

Table 1

Comparison between the calculation results and the simulation results of the resonant frequencies of V-shaped cantilevers.

$L_0(\text{mm})$	$f(\text{Hz})$	$f'(\text{Hz})$	δ
0	32.95	32.7	0.75 %
10	31.37	31.39	-0.06 %
20	29.94	30.35	-1.37 %
30	28.66	29.04	-1.33 %
40	27.59	27.88	-1.05 %
50	26.76	27.02	-0.97 %
60	26.26	26.52	-0.99 %
70	26.21	26.08	0.50 %
80	26.85	26.66	0.71 %
90	28.63	28.03	2.09 %
100	32.95	32.22	2.22 %

It can be seen that, from Table 1., a very good agreement is obtained between the calculation results and the simulation results, yielding little relative error (less than 3%). When $L_0=60 \text{ mm}$, the simulated shape is shown in Fig.5.

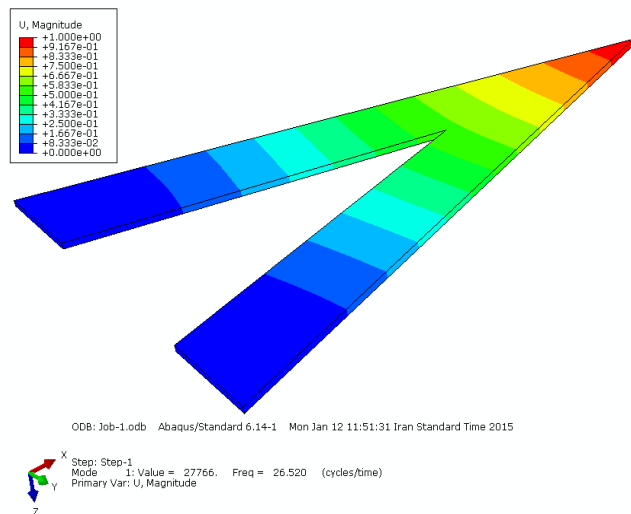


Fig.5
Deformed shaped for the first vibration mode.

4 APPLICATION

The resonant frequency formula presented in this paper is useful for many applications. First, this simple formula can be effectively used to determine the resonant frequency of V-shaped cantilevers of any dimensions and material

properties. Another significant application is the optimization of V-shaped cantilever vibration energy harvesters. The sensitivity of resonant cantilever vibration energy harvesters is directly proportional to the resonant frequency, and the resonant frequency is a key parameter to design a mechanical energy harvester. As mentioned above, with given length L_1 , given width W_1 , given thickness H and given material properties E and ρ , tapered cantilever-a special kind of V-shaped cantilever-can reach the maximum resonant frequency and highest sensitivity.

For a tapered cantilever, substituting $b=0$ into Eq. (17), the maximum resonant frequency is obtained .

$$f_{tap} = \frac{H}{2\pi L_1^2} \sqrt{\frac{70E}{\rho}} g(a,0) = \frac{H}{2\pi L_1^2} \sqrt{\frac{70E}{\rho}} \frac{\sqrt{3}}{7} = \frac{H}{2\pi L_1^2} \sqrt{\frac{30E}{7\rho}} \approx 0.3295 \frac{H}{L_1^2} \sqrt{\frac{E}{\rho}} \quad (19)$$

Apparently, the resonant frequency of a tapered cantilever is unrelated to its width W_1 . It is necessary to point out that, for a tapered cantilever, when increasing W_1 and keeping other parameters fixed, its resonant frequency will remain constant. It is worth comparing Eq. (12) and Eq. (19), and we can get the resonant frequency ratio of tapered cantilever and rectangular cantilever.

$$\frac{f_{tap}}{f_{rect}} = \frac{0.3295 \frac{H}{L_1^2} \sqrt{\frac{E}{\rho}}}{0.1639 \frac{H}{L_1^2} \sqrt{\frac{E}{\rho}}} = 2.0104 > 2 \quad (20)$$

Hence, the tapered cantilevers can lead to much higher resonant frequency and higher sensitivity than that of rectangular cantilevers.

5 CONCLUSIONS

This paper deduces a highly precise explicit formula to calculate the fundamental resonant frequency of V-shaped cantilevers based on Rayleigh-Ritz method. With this analytical formula, the calculation results are in perfect agreement with the simulation results, yielding little relative error (less than 3%). An application for calculating frequency of V-shaped cantilever energy harvesters, is presented with this formula in order to achieve a Multi-Modal energy harvester. This formula can be commonly used in the design and optimization of vibration energy harvesters. The V-shaped cantilever achieves maximum resonant frequency and highest sensitivity only when it turns into a tapered cantilever. In tapered cantilevers the resonant frequency can be doubled in comparison with simple rectangular ones.

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