Journal of Solid Mechanics Vol. 8, No. 3 (2016) pp. 590-601

Analytical and Numerical Modelling of the Axisymmetric Bending of Circular Sandwich Plates with the Nonlinear Elastic Core Material

A. Kudin *, S. Choporov ,Yu. Tamurov , M.A.V. Al Omari

Department of Mathematics, Zaporizhzhya National University, Zhukovsky street 66, Zaporizhzhya 69600, Ukraine

Received 23 June 2016; accepted 19 August 2016

ABSTRACT

Herein paper compares the analytical model with the FEM based numerical model of the axisymmetric bending of circular sandwich plates. Also, the paper describes equations of the circular symmetrical sandwich plates bending with isotropic face sheets and the nonlinear elastic core material. The method of constructing an analytical solution of nonlinear differential equations has been described. The perturbation method for differential equations with small parameters is used to represent nonlinear differential equations as a sequence of linear equations. Linear differential equations are reduced to Bessel's equation. It is compared results of analytical model with results of other researches using two problems: 1) the problem of axisymmetric transverse bending of a circular sandwich plate, 2) the problem of axisymmetric transverse bending of an annular sandwich plate. The effect of accounting nonlinear elastic core material on the strain state of the sandwich plate is described.

Keywords : Circular sandwich plate; Nonlinear elastic material; The finite element method; Perturbation method.

1 INTRODUCTION

COMPOSITE materials (composites) and layered materials are one of the great technological advances of a modern engineering. By the term , layered materials are assumed materials that are combinations of two or more organic or inorganic layers. Layered materials allow to optimize some physical and mechanical properties of constructions. Sandwich structures are widely used in the aircraft and shipbuilding industries, the aerospace industry, civil engineering, electronics and other industries. Thus the stress-strain state analysis of sandwich structural elements is urgent. Herein study has investigated by analytical and numerical methods. Currently, there are many experimental and theoretical works devoted to sandwich structures, for example [1-11]. Well- known articles reviews devoted to sandwich structures, for example [1,7]. These works presented linear elastic models for core material. On the other hand, for some materials (e.g. cooper, duralumin, aluminium bronze, composites [12]) linear elastic models do not accurately describe the observed material behavior. Thus, bending of circular sandwich plates with nonlinear elastic core still less investigated. This paper has derived a nonlinear differential equations of symmetric sandwich plates bending with isotropic face sheets and nonlinear elastic core material by [12] (the analytical model). Using the perturbation method for differential equations with small parameters we reduce

*Corresponding author. Tel.: +38 0612 287614.
E-mail address: avk256@gmail.com (A.Kudin).

nonlinear differential equations to linear systems. Linear differential equations are reduced to Bessel's equation. Numerical results of the analytical model are compared with [2, 6, 8] and with the finite elements model.

In numerical analysis of solids, two main approaches have been adopted to simulate layered materials [13]:

1. Techniques, which explicitly model the discontinuous nature of the material. This approach includes the Discontinuous Deformation (Displacement) Analysis (DDA) [14] and the Discrete Element Method (DEM) [15, 16]. The Finite Element Method (FEM) or the Finite Difference Method (FDM), which utilizes layers topology, interfaces or contact technology, also included in the first approach. The Discontinuous Deformation Analysis and the Discrete Element Method are common used in rock mechanics, simulation of granular materials and the micro-dynamics of powder flows. These techniques provide a more accurate description of layered material nature.
2. Techniques, which use equivalent continuum model of the material. This approach includes the FEM and the FDM equivalent continuum model. In the equivalent continuum technique the discrete material is replaced by a homogeneous continuum. e.g., in Cosserat theory, one of the mathematical models describing the mechanics of general micropolar continua, each point of the continuum is associated with independent rotational degrees of freedom in addition to translational degrees of freedom. The basic kinematics variables of Cosserat theory are the displacements, the first-order displacement gradients, the microstructural rotations and the rotation gradients. Higher-order displacement gradients are not considered [13].

In the numerical example we use FEM, which utilizes layers topology: obtained meshes approximate borders of layers by edges and faces of finite elements.

The main objective of herein paper is to develop analytical model for stress-strain state of circular and annular sandwich plates with nonlinear elastic core. Analytical model shows difference between linear elastic model and nonlinear elastic model of the core. Also, it has developed numerical finite element model and compare analytical solutions with numerical solutions. This comparison demonstrates the accuracy of both models presented in the paper.

2 THE STATE OF STRESS OF A PLATE

Consider a circular sandwich plate that is loaded by a transverse load $q(r,\varphi)$. Face sheets are δ_1 and δ_2 thick correspondingly. Face sheets are made from isotropic material for which Hooke's law is a useful approximation. A core is $2h$ thick (Fig. 1). It is made from nonlinear elastic isotropic material.

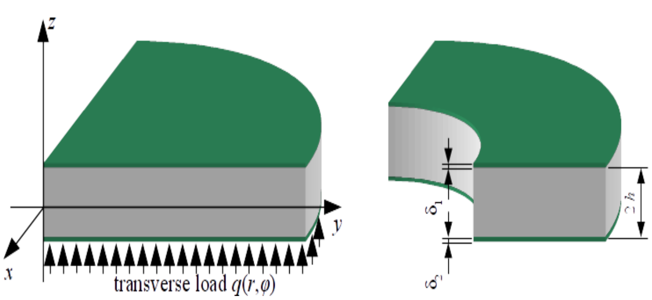

Fig.1
Circular and annular sandwich plates.

Face sheets correspond to Kirchhoff plate theory. The state of stress in these layers is determined by Hooke's law [17, 18]

$$\begin{aligned}
&h \le z \le h+\delta_1: \\
&\sigma_r = \frac{E}{\left(1-\mu^2\right)}\left(\varepsilon_r + \mu\varepsilon_\varphi\right), \sigma_\varphi = \frac{E}{\left(1-\mu^2\right)}(\varepsilon_\varphi + \mu\varepsilon_r), \tau_{r\varphi} = G\,\varepsilon_{r\varphi} \\
&-h-\delta_2 \le z \le -h: \\
&\sigma_r^* = \frac{E}{\left(1-\mu^2\right)}\left(\varepsilon_r^* + \mu\varepsilon_\varphi^*\right), \sigma_\varphi^* = \frac{E}{\left(1-\mu^2\right)}(\varepsilon_\varphi^* + \mu\varepsilon_r^*), \tau_{r\varphi}^* = G^*\varepsilon_{r\varphi}^*
\end{aligned} \tag{1}$$

where E is Young's modulus, G is the shear modulus, and μ is Poisson's ratio for the material of the layer $h \le z \le h+\delta_1$. Index * in Eq. (1) refers to mechanical properties of the layer $-h-\delta_2 \le z \le -h$. The state of stress in the core is determined by expressions [12]

$$\begin{aligned}
&\tilde{\sigma}_r = 3\tilde{K}(1+\chi_2\varepsilon_0^2)\varepsilon_0 + 2\tilde{G}(1+\gamma_2\psi_0^2)(\tilde{\varepsilon}_r - \varepsilon_0),\\
&\tilde{\sigma}_\varphi = 3\tilde{K}(1+\chi_2\varepsilon_0^2)\varepsilon_0 + 2\tilde{G}(1+\gamma_2\psi_0^2)(\tilde{\varepsilon}_\varphi - \varepsilon_0),\\
&\tilde{\tau}_{r\varphi} = \tilde{G}(1+\gamma_2\psi_0^2)\tilde{\varepsilon}_{r\varphi}, \tilde{\tau}_{rz} = \tilde{G}(1+\gamma_2\psi_0^2)\tilde{\varepsilon}_{rz}, \tilde{\tau}_{\varphi z} = \tilde{G}(1+\gamma_2\psi_0^2)\tilde{\varepsilon}_{\varphi z}.
\end{aligned} \tag{2}$$

In Eq. (2), the shear modulus is denoted by $\tilde{G}$ and the bulk modulus is denoted by $\tilde{K}$: ε_0 is elongation, ψ_0^2 is the strain intensity factor

$$\begin{aligned}
&\varepsilon_0 = \frac{1}{3}\left(\tilde{\varepsilon}_r + \tilde{\varepsilon}_\varphi + \tilde{\varepsilon}_z\right), \psi_0^2 = \frac{8}{9}\left(\tilde{\varepsilon}_r^2 + \tilde{\varepsilon}_\varphi^2 + \tilde{\varepsilon}_z^2 - \tilde{\varepsilon}_r\tilde{\varepsilon}_\varphi - \tilde{\varepsilon}_\varphi\tilde{\varepsilon}_z - \tilde{\varepsilon}_z\tilde{\varepsilon}_r\right) + \frac{2}{3}\left(\tilde{\varepsilon}_{r\varphi}^2 + \tilde{\varepsilon}_{rz}^2 + \tilde{\varepsilon}_{\varphi z}^2\right),\\
&\tilde{\varepsilon}_z = -\frac{\tilde{\mu}}{1-\tilde{\mu}}\left(\tilde{\varepsilon}_r + \tilde{\varepsilon}_\varphi\right), \tilde{\mu} = \frac{1}{2}\frac{3\tilde{K} - 2\tilde{G}}{3\tilde{K} + \tilde{G}}.
\end{aligned}$$

Parameter γ_2 characterizes shape changes of the structural element in the nonlinear elastic deformation stage. This parameter is determined experimentally [12]; parameter χ_2 characterizes the volume element changes.

3 STRAIN-DISPLACEMENT RELATIONS FOR SANDWICH PLATES

The state of strain at some midplane point of the sandwich plate face sheets is determined by radial displacements $u_i(r,\varphi)$, angular displacements $v_i(r,\varphi), (i=1,2)$, and deflection $w(r,\varphi)$.

By adopted hypotheses, we have [10]:

$$\begin{cases}
h \le z \le h+\delta_1: & u(r,\varphi,z) = u_1(r,\varphi) + \left(z - h - \dfrac{\delta_1}{2}\right)w_{,r},\\
& v(r,\varphi,z) = v_1(r,\varphi) + \left(z - h - \dfrac{\delta_1}{2}\right)\dfrac{w_{,\varphi}}{r};\\
-h-\delta_2 \le z \le -h: & u^*(r,\varphi,z) = u_2(r,\varphi) + \left(z + h + \dfrac{\delta_2}{2}\right)w_{,r},\\
& v^*(r,\varphi,z) = v_2(r,\varphi) + \left(z + h + \dfrac{\delta_2}{2}\right)\dfrac{w_{,\varphi}}{r};\\
-h \le z \le h: & \tilde{u}(r,\varphi,z) = u_1(r,\varphi)a_1(z) + u_2(r,\varphi)a_2(z) + c_0^*(z)w_{,r},\\
& \tilde{v}(r,\varphi,z) = v_1(r,\varphi)b_1(z) + v_2(r,\varphi)b_2(z) + c_0^{**}(z)\dfrac{w_{,\varphi}}{r}.
\end{cases} \tag{3}$$

Using [10] in Eq. (3), we obtain

$$\begin{aligned}
&a_1(z) = 1+\alpha_2(z), a_2(z) = -\alpha_2(z), c_0^*(z) = z - h - \frac{\delta_1}{2} - \alpha\alpha_2(z), c_0^{**}(z) = \lambda_3(z) - \alpha\alpha_3(z),\\
&b_1(z) = 1+\alpha_3(z), b_2(z) = -\alpha_3(z), \alpha = 2h + \frac{\delta_1+\delta_2}{2}, \alpha_2(z) = \frac{\Phi_1(z) - \Phi_1(h)}{\Phi_1(h) - \Phi_1(-h)},\\
&\alpha_3(z) = \frac{\Phi_2(z) - \Phi_2(h)}{\Phi_2(h) - \Phi_2(-h)}, \Phi_1(z) = \frac{1}{\tilde{G}}\int f_1(z)dz, \Phi_2(z) = \frac{1}{\tilde{G}}\int f_2(z)dz.
\end{aligned}$$

Functions $f_1(z), f_2(z)$ are transverse shear-stress distribution functions for stresses $\tilde{\tau}_{rz}$ and $\tilde{\tau}_{\varphi z}$ in the thickness direction. Strains are represented by [12].

The face sheet layer $h \le z \le h+\delta_1$:

$$\varepsilon_r = u_{,r}, \varepsilon_\varphi = \frac{1}{r}v_{,\varphi} + \frac{u}{r}, \varepsilon_{r\varphi} = \frac{1}{r}u_{,\varphi} + v_{,r} - \frac{v}{r}$$

Another face sheet layer $-h-\delta_2 \le z \le -h$:

$$\varepsilon_r^* = u_{,r}^*, \varepsilon_\varphi^* = \frac{1}{r}v_{,\varphi}^* + \frac{u^*}{r}, \varepsilon_{r\varphi}^* = \frac{1}{r}u_{,\varphi}^* + v_{,r}^* - \frac{v^*}{r}$$

and the core layer $-h \le z \le h$:

$$\tilde{\varepsilon}_r = \tilde{u}_{,r}, \tilde{\varepsilon}_\varphi = \frac{1}{r}\tilde{v}_{,\varphi} + \frac{\tilde{u}}{r}, \tilde{\varepsilon}_{r\varphi} = \frac{1}{r}\tilde{u}_{,\varphi} + \tilde{v}_{,r} - \frac{\tilde{v}}{r}, \tilde{\varepsilon}_{rz} = \tilde{u}_{,z} + w_{,r}, \tilde{\varepsilon}_{\varphi z} = \tilde{v}_{,z} + \frac{1}{r}w_{,\varphi}.$$

The total potential energy of the circular sandwich plate is

$$\mathrm{E} = \frac{1}{2}\iint\left(\int_h^{h+\delta_1} \sigma_r\varepsilon_r + \sigma_\varphi\varepsilon_\varphi + \tau_{r\varphi}\varepsilon_{r\varphi}dz + \int_{-h-\delta_2}^{-h} \sigma_r^*\varepsilon_r^* + \sigma_\varphi^*\varepsilon_\varphi^* + \tau_{r\varphi}^*\varepsilon_{r\varphi}^*dz + \right.$$
$$\left. + \int_{-h}^{h} \tilde{\sigma}_r\tilde{\varepsilon}_r + \tilde{\sigma}_\varphi\tilde{\varepsilon}_\varphi + \tilde{\tau}_{r\varphi}\tilde{\varepsilon}_{r\varphi} + \tilde{\tau}_{rz}\tilde{\varepsilon}_{rz} + \tilde{\tau}_{\varphi z}\tilde{\varepsilon}_{\varphi z}dz - qw\left(r,\varphi\right)\right) rdrd\varphi = 0$$

Differential equations of equilibrium are obtained from total potential energy by applying the principle of possible movements [9].

4 EQUILIBRIUM EQUATIONS FOR AXISYMMETRIC BENDING

Consider axisymmetric transverse bending of a circular sandwich plate. Now suppose that the transverse load $q(r)$ is axisymmetric and uniform. In addition, suppose that the plate have face sheets with equal thickness $\delta_1 = \delta_2 = \delta$.

Using the symmetry of the strain state of the plate and taking into account the absence of any angular displacement ($u_1 = -u_2 = u, v_1 = -v_2 = v = 0$), we obtain nonlinear differential equations of equilibrium

$$\begin{aligned} &B_{11}w_{,rrr} + B_{12}w_{,rr} + B_{13}u_{,rr} + B_{14}w_{,r} + B_{15}u_{,r} + B_{16}u\left(r\right) + \Phi_1 = 0, \\ &B_{21}w_{,rrrr} + B_{22}w_{,rrr} + B_{23}u_{,rrr} + B_{24}w_{,rr} + B_{25}u_{,rr} + +B_{26}w_{,r} + B_{27}u_{,r} + B_{28}u\left(r\right) - rq + \Phi_2 = 0. \end{aligned} \tag{4}$$

Nonlinear terms of the Eqs. (4) Φ_1, Φ_2 and the coefficients presented in [3] become very cumbersome. Boundary conditions were considered by A.P. Prusakov in [8].

Simply-supported edge:

$$w(r) = 0, w_{,rr} + \frac{\mu}{r}w_{,r} = 0, u_{,r} + \frac{\mu}{r}u(r) = 0, r = R. \tag{5}$$

Clamped or fixed edge:

$$w(r) = 0, w_{,r} = 0, u(r) = 0, r = R. \tag{6}$$

Now if we recall (4) with boundary conditions (5) or (6) we obtain the analytical model.

Perturbation method are widely used for solving differential equations [12, 19]. To solve these equations we use approximation techniques in terms of a small parameter. The small parameter appears naturally in the equations or may be artificially introduced for convenience. Thus we use perturbation method for simplifying nonlinear differential equations.

Let the naturally small parameter in Eqs. (4) be a ratio of mechanical properties of the core

$$\lambda = \frac{\gamma_2}{\left(3\tilde{K} + 4\tilde{G}\right)^3}$$

Displacements are represented by series

$$u(r) = u_0(r) + \sum_{i=1}^{n} \lambda^i u_i(r); w(r) = w_0(r) + \sum_{i=1}^{n} \lambda^i w_i(r) \tag{7}$$

The solutions of Eqs. (4) are represented by the first few terms of an asymptotic expansions (7), usually not more than two or three terms. In present paper we use three terms of series (7).

Substituting (7) in (4) and collecting coefficients in terms of the powers of λ we obtain a system of equations for zeroth-order and higher-ordered approximations.

We have the system of equations of the zeroth-order approximation (linear elastic formulation of the problem)

$$\begin{aligned} &B_{11} w_{0,rrr} + B_{12} w_{0,rr} + B_{13} u_{0,rr} + B_{14} w_{0,r} + B_{15} u_{0,r} + B_{16} u_0(r) = 0, \\ &B_{21} w_{0,rrrr} + B_{22} w_{0,rrr} + B_{23} u_{0,rrr} + B_{24} w_{0,rr} + B_{25} u_{0,rr} + +B_{26} w_{0,r} + B_{27} u_{0,r} + B_{28} u_0(r) - rq = 0 \end{aligned} \tag{8}$$

System of equations for the first-order approximations

$$\begin{aligned} &B_{11} w_{1,rrr} + B_{12} w_{1,rr} + B_{13} u_{1,rr} + B_{14} w_{1,r} + B_{15} u_{1,r} + B_{16} u_1(r) = \Psi_{11}, \\ &B_{21} w_{1,rrrr} + B_{22} w_{1,rrr} + B_{23} u_{1,rrr} + B_{24} w_{1,rr} + B_{25} u_{1,rr} + +B_{26} w_{1,r} + B_{27} u_{1,r} + B_{28} u_1(r) = \Psi_{21} \end{aligned}$$

where

$$\begin{aligned} \Psi_{11} = {} & B^1_{110} u_0(r) w_{0,r} u_{0,r} + B^1_{111} u_0(r)^3 + B^1_{112} u^3_{0,r} + B^1_{113} w^3_{0,r} + B^1_{114} w^2_{0,rr} w_{0,rrr} + B^1_{115} w^2_{0,rr} u_{0,rr} + B^1_{116} u^2_{0,r} w_{0,rrr} + \\ & + B^1_{117} w_{0,rrr} w_{0,r} + B^1_{118} u_0(r) w^2_{0,r} + B^1_{119} w^2_{0,r} u_{0,r} + B^1_{120} u^2_{0,r} u_{0,r} + B^1_{121} w_{0,r} u^2_{0,r} + B^1_{122} u_0(r)^2 u_{0,r} + B^1_{123} u_0(r)^2 w_{0,r} + \\ & + B^1_{124} u_{0,r} w_{0,rr} w_{0,rrr} + B^1_{125} u_{0,r} u_{0,rr} w_{0,rr} + B^1_{126} w_{0,rr} u_{0,r} u_0(r) + B^1_{127} u_{0,rr} w_{0,r} u_0(r) + B^1_{128} u_0(r)^2 w_{0,rrr} + \\ & + B^1_{129} w_{0,r} w_{0,rr} w_{0,rrr} + B^1_{130} u_0(r) u_{0,rr} w_{0,rr} + B^1_{131} u_0(r) w_{0,rr} w_{0,rrr} + B^1_{132} u_{0,r} w_{0,r} u_{0,rr} + B^1_{133} u_0(r) u_{0,r} w_{0,rrr} + \\ & + B^1_{134} w_{0,rr} w_{0,r} u_{0,r} + B^1_{135} u_0(r) w_{0,r} w_{0,rrr} + B^1_{136} w_{0,r} u_{0,rr} w_{0,rr} + B^1_{137} w_{0,rr} w_{0,r} u_0(r) + B^1_{138} w^2_{0,rr} w_{0,r} + \\ & + B^1_{139} w^2_{0,rr} u_{0,r} + B^1_{140} u^2_{0,rr} u_{0,rr} + B^1_{141} w_{0,rr} w^2_{0,r} + B^1_{142} u^2_{0,r} w_{0,rr} + B^1_{143} w_{0,rrr} w^2_{0,r} + B^1_{144} w^2_{0,rr} u_0(r) + \\ & + B^1_{145} u_0(r) u_{0,r} u_{0,rr} + B^1_{146} u_{0,rr} w^2_{0,r} + B^1_{147} u_0(r)^2 u_{0,rr} + B^1_{148} w^3_{0,rr} + B^1_{149} w_{0,rr} u_0(r)^2, \end{aligned}$$

$$\Psi_{21} = rq + B_{210}^{1}u_0(r)^3 + B_{211}^{1}u_{0,r}^{3} + B_{212}^{1}w_{0,r}^{3} + B_{213}^{1}u_0(r)w_{0,r}u_{0,r} + B_{214}^{1}w_{0,rr}u_0(r)^2 + B_{215}^{1}w_{0,rrr}^{2}u_0(r) +$$

$$+B_{216}^{1}u_{0,rr}^{2}w_{0,r} + B_{217}^{1}u_{0,rr}^{2}u_0(r) + B_{218}^{1}w_{0,rrr}^{2}w_{0,r} + B_{219}^{1}w_{0,rr}w_{0,r}^{2} + B_{220}^{1}w_{0,rr}w_{0,r}^{2} + B_{221}^{1}u_{0,r}^{2}w_{0,rr} +$$

$$+B_{222}^{1}u_0(r)^2u_{0,rr} + B_{223}^{1}w_{0,rr}^{3} + B_{224}^{1}u_{0,r}w_{0,rr}u_{0,rrr} + B_{225}^{1}u_{0,r}w_{0,rr}w_{0,rrrr} + B_{226}^{1}u_{0,rr}w_{0,rr}w_{0,rrr} +$$

$$+B_{227}^{1}u_{0,r}u_{0,rr}w_{0,rrr} + B_{228}^{1}u_0(r)w_{0,rr}u_{0,rrr} + B_{229}^{1}w_{0,r}w_{0,rr}w_{0,rrrr} + B_{230}^{1}u_0(r)w_{0,rr}w_{0,rrrr} +$$

$$+B_{231}^{1}w_{0,r}u_{0,rr}w_{0,rrr} + B_{232}^{1}u_{0,r}w_{0,r}u_{0,rrr} + B_{233}^{1}u_0(r)u_{0,rr}w_{0,rrr} + B_{234}^{1}u_0(r)u_{0,r}w_{0,rrrr} + B_{235}^{1}u_0(r)^2w_{0,rrr} +$$

$$+B_{236}^{1}w_{0,rr}^{2}w_{0,rrr} + B_{237}^{1}w_{0,rr}^{2}u_{0,rrr} + B_{238}^{1}u_{0,r}^{2}w_{0,rrr} + B_{239}^{1}w_{0,rrr}w_{0,r}^{2} + B_{240}^{1}w_{0,rr}^{2}u_0(\mathrm{r}) + B_{241}^{1}w_{0,rr}^{2}w_{0,r} +$$

$$+B_{242}^{1}w_{0,rr}^{2}u_{0,r} + B_{243}^{1}u_{0,rr}w_{0,r}^{2} + B_{244}^{1}u_{0,r}^{2}u_{0,rr} + B_{245}^{1}u_{0,rr}w_{0,r}u_0(r) + B_{246}^{1}u_{0,r}w_{0,r}u_{0,rr} + B_{247}^{1}u_{0,r}u_{0,rr}u_0(r) +$$

$$+B_{248}^{1}w_{0,rr}w_{0,r}u_0(r) + B_{249}^{1}w_{0,rr}u_{0,r}u_0(r) + B_{250}^{1}w_{0,rrr}w_{0,r}^{2} + B_{251}^{1}u_{0,rrr}u_0(r)^2 + B_{252}^{1}w_{0,rrr}u_0(r)^2 +$$

$$+B_{253}^{1}u_{0,rrr}w_{0,r}^{2} + B_{254}^{1}u_{0,rrr}u_{0,r}u_0(r) + B_{255}^{1}w_{0,r}w_{0,rr}u_{0,rrr} + B_{256}^{1}u_{0,r}u_{0,rr}w_{0,rr} + B_{257}^{1}w_{0,rrr}w_{0,r}u_0(r) +$$

$$+B_{258}^{1}w_{0,rrr}w_{0,r}u_{0,r} + B_{259}^{1}w_{0,rr}u_{0,rr}w_{0,r} + B_{260}^{1}w_{0,rrrr}w_{0,r}u_0(r) + B_{261}^{1}u_{0,rrr}w_{0,r}u_0(r) +$$

$$+B_{262}^{1}w_{0,rrr}w_{0,rr}w_{0,r} + B_{263}^{1}u_{0,rrr}u_{0,r}^{2} + B_{264}^{1}u_{0,rr}^{2}u_{0,r} + B_{265}^{1}w_{0,rr}^{2}w_{0,rrr} + B_{266}^{1}w_{0,rrr}^{2}w_{0,rr} + B_{267}^{1}u_{0,rr}^{2}w_{0,rr} +$$

$$+B_{268}^{1}w_{0,rrr}^{2}u_{0,r} + B_{269}^{1}u_{0,r}^{2}w_{0,rrr} + B_{270}^{1}w_{0,rr}^{2}u_{0,rrr} + B_{271}^{1}w_{0,rr}u_{0,rr}u_0(r) + B_{272}^{1}w_{0,rrr}w_{0,rr}u_0(r) +$$

$$+B_{273}^{1}w_{0,rrr}u_{0,r}u_0(r) + B_{274}^{1}w_{0,rr}w_{0,r}u_{0,r} + B_{275}^{1}w_{0,rrr}w_{0,rr}u_{0,r} + B_{276}^{1}w_{0,r}^{2}u_0(r) + B_{277}^{1}u_{0,r}^{2}w_{0,r} +$$

$$+B_{278}^{1}u_{0,r}u_0(r)^2 + B_{279}^{1}w_{0,r}^{2}u_{0,r} + B_{280}^{1}u_{0,r}^{2}u_0(r) + B_{281}^{1}w_{0,r}u_0(r)^2.$$

The coefficients $B_{110}^{1} - B_{149}^{1}$, $B_{210}^{1} - B_{281}^{1}$ are related to mechanical properties of the plate.
Now, using [3], we get higher-order approximations

$$\begin{aligned} &B_{11}w_{i,rrr} + B_{12}w_{i,rr} + B_{13}u_{i,rr} + B_{14}w_{i,r} + B_{15}u_{i,r} + B_{16}u_i(r) = \Psi_{1i}, \\ &B_{21}w_{i,rrrr} + B_{22}w_{i,rrr} + B_{23}u_{i,rrr} + B_{24}w_{i,rr} + B_{25}u_{i,rr} + +B_{26}w_{i,r} + B_{27}u_{i,r} + B_{28}u_i(r) = \Psi_{2i} \end{aligned} \tag{9}$$

In Eq. (9), Ψ_{1i} and Ψ_{2i} are nonlinear terms for i -order approximations. Now we can consider a sequence of linear equations instead nonlinear system (4).

5 SOLVE A SYSTEM OF DIFFERENTIAL EQUATIONS

Now, differential Eqs. (8) will be represented by operators

$$\begin{aligned}
&L_1\left(-\frac{\delta}{2}B_{151}w_{0,r}+B_{15}u_0\right)+L_3\left(B_{141}w_{0,r}+\frac{2}{2h-\delta}B_{141}u_0\right)=0,\\
&L_2\left(-\frac{\delta^2}{6}(3B_{151}+B_{152})w_{0,r}+\delta B_{151}u_0\right)++L_4\left(-B_{141}(2h-\delta)w_{0,r}+(-2)B_{141}u_0\right)=rq
\end{aligned} \tag{10}$$

where by L_1, L_2, L_3 and L_4 denoted differential operators that are defined for arbitrary function $g(r)$. Therefore,

$$\begin{aligned}
&L(g)=\left(\frac{1}{r}(rg)_{,r}\right)_{,r}=g_{,rr}+\frac{g_{,r}}{r}-\frac{g}{r^2}, L_1(g)=rL(g)=rg_{,rr}+g_{,r}-\frac{g}{r},\\
&L_2(g)=\left(L_1(g)\right)_{,r}=rg_{,rrr}+2g_{,rr}-\frac{g_{,r}}{r}+\frac{g}{r^2}, L_3(g)=rg, L_4(g)=\left(L_3(g)\right)_{,r}=rg_{,r}+g
\end{aligned}$$

Fist we integrate the Eq. (10); secondly, we add the first equation to the second. Now by simplifying, we obtain the modified Bessel differential equation

$$L\left(u_0\right)-\beta^2u_0=\eta_0(r) \tag{11}$$

where

$$\begin{aligned}
&\beta^2=-B_{141}\left(\frac{2}{2h-\delta}-\frac{D_2}{D_1}\right)\frac{2D_1}{B_{15}2D_1+\delta B_{151}D_2}, \eta_0(r)=\left(L\left(\frac{\delta B_{151}}{2D_1}\xi_0(r)\right)-\frac{B_{141}}{D_1}\xi_0(r)\right)\frac{2D_1}{B_{15}2D_1+\delta B_{151}D_2},\\
&D_1=\left(-\delta hB_{151}-\frac{\delta^2}{6}B_{152}\right), D_2=\left(2(h-\delta)B_{151}+\left(2h-\delta\right)B_{152}\right),\\
&\xi_0(r)=\frac{1}{r}\int\left(r\int\left(\frac{qr}{2}+\frac{C_1}{r}\right)dr\right)dr=\frac{qr^3}{16}+\frac{1}{4}C_1r(2\ln(r)-1)+\frac{1}{2}C_2r+\frac{1}{r}C_3.
\end{aligned}$$

Integrating (11) in u_0, we get

$$u_0=C_4I_1(\beta r)+C_5K_1(\beta r)-K_1(\beta r)\int I_1(\beta r)\eta_0(r)rdr+I_1(\beta r)\int K_1(\beta r)\eta_0(r)rdr, w_0=\int\frac{\xi(r)-D_2u_0}{D_1}dr.$$

Similarly, we obtain the solution of (9). In this case, functions $\xi_0(r)$ and $\eta_0(r)$ that depend on the right-hand side of (9).

6 FINITE ELEMENT MODEL

Consider the finite element model of a sandwich plate. We may assume that the sandwich plate is the thin three-dimensional solid. Thus we can explicitly model the discontinuous nature of the material. In this case, obtained meshes utilize layers topology. Borders of layers is approximated by edges and faces of finite elements (see Fig. 2). Obtained meshes can be block-structured or unstructured. In both cases, for each layer of material we generate layers of elements (each layer of material needs at least one element along the thickness of the layer).

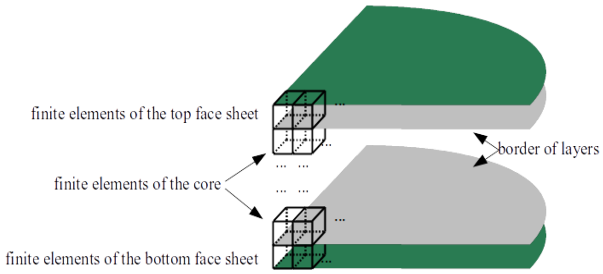

Fig.2
Finite elements layers.

Let the finite elements mesh *M* consist of two subsets of elements

$$M = M_f \cup M_c, M_f \cup M_c \equiv \varnothing$$

where M_f is the set of face sheets finite elements; M_c is the set of the core finite elements.

For three-dimensional stress and strain we get the element stiffness equation as:

$$[k_m] = \iiint [B_m]^T [D_m][B_m] dxdydz$$

where for 8-node hexahedra element *m* we get

$$[B_m] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} H_1 & H_2 & \dots & H_8 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & H_1 & H_2 & \dots & H_8 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & H_1 & H_2 & \dots & H_8 \end{bmatrix}$$

Here $H_i, (i = \overline{1,8})$ is a standard three-dimensional shape function ($H_i(\xi,\eta,\zeta)$ for isoparametric hexahedra). Elasticity matrix $[\mathrm{D}_m]$ depends on properties of layer's material. Now we get

$$[D_m] = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1 & \frac{\mu}{1-\mu} & \frac{\mu}{1-\mu} & 0 & 0 & 0 \\ \frac{\mu}{1-\mu} & 1 & \frac{\mu}{1-\mu} & 0 & 0 & 0 \\ \frac{\mu}{1-\mu} & \frac{\mu}{1-\mu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2(1-\mu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2(1-\mu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2(1-\mu)} \end{bmatrix}$$

where E_m is Young's modulus, μ_m is Poisson's ratio of the element *m*. For sandwich plate

$$E_m = \begin{cases} E, & if \ \ m \in M_f, \\ \tilde{E}, & if \ \ m \in M_c, \end{cases} \qquad \mu_m = \begin{cases} \mu, & if \ \ m \in M_f, \\ \tilde{\mu}, & if \ \ m \in M_c. \end{cases}$$

Here E is Young's modulus of face sheets, μ is Poisson's ratio of face sheets; $\tilde{E}$ is Young's modulus of the core, $\tilde{\mu}$ is Poisson's ratio of the core.

Remark. A full three-dimensional finite element modelling is computationally expensive. Meshes with good aspect ratio of hexahedrons require a large number of elements (e.g. 3-D model of the circular sandwich plate has more than 200,000 nodes).

7 NUMERICAL EXAMPLES AND VALIDATION TEST

7.1 Circular sandwich plates with elastic core

Consider a circular sandwich plate with elastic core; the plate loaded transversely by a distributed load q. Suppose the core thickness $2h$ is equal to $16 \cdot 10^{-3} m$; face sheets of equal thickness: $\delta_1 = \delta_2 = \delta = 1 \cdot 10^{-3} m$; the radius of the plate R is equal to 0.4 m; the shear modulus of the core $\tilde{G}$ is equal to $2.77 \cdot 10^4 Mpa$ and the bulk modulus of the core $\tilde{K}$ is equal to $6 \cdot 10^4 Mpa$; the shear modulus of face sheets G is equal to $8 \cdot 10^4 Mpa$ and the Poisson's ratio of face sheets μ is equal to 0.27 . The tensile properties of the core are linear in transverse directions.

Table 1. compares the maximum deflection in the center of the sandwich plate with elastic core.

Table 1
Transverse bending of the circular sandwich plate with elastic core.

q, Mpa	$w_{\max} = w(0), 10^{-3} m$								
	Boundary conditions								
	Simply-supported Edge				Clamped Edge				
	Model				Model				
	1	2	3	4	1	2	3	4	5
0.05	1.586	1.383	1.430	2.621	0.385	0.339	0.348	0.635	0.291
0.07	2.221	1.937	2.001	3.670	0.539	0,474	0.487	0.889	0.407
0.09	2.855	2.490	2.573	4.718	0.693	0.610	0.626	1.142	0.523
0.11	3.489	3.043	3.145	5.767	0.847	0.745	0.765	1.396	0.639

Models are denoted by numerals 1-5 as follows: 1 – the analytical model; 2 – the finite element model; 3 – model considered in the paper [6]; 4 – model considered in the paper [8]; 5 – model considered in the paper [2].

7.2 Annular sandwich plates with elastic core

In this subsection we consider a sandwich annular plate with elastic core; the plate loaded transversely by a distributed load q. The annular plate has an external radius R (R is equal to 0.4 m) and internal radius b (b is equal to 0.2 m). Other properties of the plate are the same as subsection 7.1.

Table 2. compares the maximum deflection of the sandwich annular plate with free internal edge and simply-supported or clamped external edge. Conversely, Table 3. compares the maximum deflection of the sandwich plate with simply-supported or clamped internal edge and free external edge.

Table 2
Transverse bending of sandwich annular plate with free internal edge.

q, Mpa	$w_{\max} = w(b), 10^{-3} m$			
	Simply-supported Edge		Clamped Edge	
	Model		Model	
	Analytical	FEM	Analytical	FEM
0.005	0.151	0.134	0.0129	0.0114
0.007	0.212	0.187	0.0181	0.0159
0.009	0.273	0.241	0.0233	0.0204
0.011	0.333	0.294	0.0284	0.0249

Table 3
Transverse bending of sandwich annular plate with free external edge.

q, Mpa	$w_{max} = w(R), 10^{-3} m$			
	Simply-supported Edge		Clamped Edge	
	Model		Model	
	Analytical	FEM	Analytical	FEM
0.005	0.199	0.176	0.0218	0.0188
0.007	0.279	0.247	0.0305	0.0264
0.009	0.359	0.317	0.0393	0.0339
0.011	0.439	0.387	0.0480	0.0411

7.3 Validation test

Finite element model validation is the verification that idealization premises and analysis conclusions are valid. Thus it has compared analytical solutions with numerical finite element methods in Tables 1., 2 and 3. This comparison demonstrates the accuracy of both approaches presented in herein paper.

Next, the influence of the mesh size on the results of the numerical simulation will be examined. Element size in the thickness direction should be less than or equal to corresponding layer thickness. Consider mapped mesh for the annular plate with clamped external edge (Table 2.). Fig. 3 shows influence of the mesh size on the maximum bending of the plate under transversal load $q = 0.005 Mpa$.

Fig. 3 shows that number of finite elements increases from the left to right (the mesh size decreases). Mapped meshes for the annular plate with the small number of elements have big aspect ratio of element edges. We can get cube-like finite elements (element edges aspect ratio is close to 1) in meshes with the big number of elements (e.g. using mesh with 140000 element we get that maximum bending is $0.0115 \cdot 10^{-3} m$, as we can see numerical solution tends to this value). Thus an increase in the number of finite elements in the mesh facilitates getting to more reliable result of finite-element modelling.

Remark. It has obtained similar dependences for the influence of the mesh size on the result of the numerical simulation for other cases of boundary conditions and for circular plates.

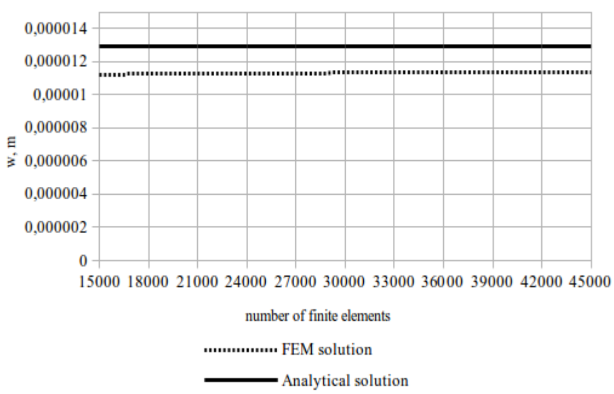

Fig.3
Influence of the mesh size on maximum bending of clamped annular plate.

8 INFLUENCE OF NONLINEARITY: NUMERICAL INVESTIGATION

Now consider a circular sandwich plate with nonlinear elastic core. Geometrical and mechanical properties are the same in the previous case. Let χ_2 is equal to 0 and γ_2 is equal to $-3.878 \cdot 10^5$.

Table 4. compares the maximum deflection in the center of the sandwich plate with nonlinear elastic core for zeroth-order and higher approximations.

Table 4
Transverse bending of the circular sandwich plate with nonlinear elastic core.

q, Mpa	$w_{\max} = w(0), 10^{-3} m$							
	Boundary conditions							
	Simply-supported Edge				Clamped Edge			
	Approximation				Approximation			
	0	1	2	3	0	1	2	3
0.05	1.586	1.598	1.598	1.598	0.385	0.389	0.389	0.389
0.07	2.221	2.253	2.254	2.254	0.539	0.550	0.551	0.551
0.09	2.855	2.924	2.929	2.929	0.693	0.717	0.718	0.718
0.11	3.489	3.615	3.628	3.630	0.847	0.891	0.892	0.892

Remark. Numerical finite element model has developed with idealization premises for linear elastic model of core material. Numerical finite solutions for nonlinear elastic models of core material will be presented in the future researches.

9 CONCLUSIONS

As the result it has implemented the analytical model and the finite element model for stress-strain state analysis of circular and annular plates. The analytical model uses the perturbation method for differential equations with small parameters to represent nonlinear differential equations as a sequence of linear equations; further, linear differential equations are reduced to Bessel's equation. The finite element model is a full three-dimensional model; this model explicitly utilize layers topology.

As shown in Tables 1, 2, 3, difference between the analytical model and the finite element model is 13-16%. The difference between the analytical model and other models is 11-40%; respectively, between the finite element model and other models is 3-47% (Table 1.). Note that these models have various simplifying assumptions.

In addition, it is compared deflection in the center of the nonlinear elastic core sandwich plate with deflection of the linear elastic core sandwich plate. The difference between linear deflection and nonlinear deflection is 1-5%.

Nonlinear dynamics and stability of sandwich structural elements are suggested as opportunities for future researches.

REFERENCES

[1] Carrera E., 2003, Historical review of zig-zag theories for multilayered plates and shells, *Applied Mechanics Reviews* **56**:287-308.

[2] Gorshkov A.G., Starovoitov E.I., Yarovaya A.V., 2005, *Mechanics of Layer Viscoelastoplastic Construction Elements*, Fizmatlit, Moscow.

[3] Kudin A.V., Tamurov Yu.N., 2011, Modeling of bending symmetric sandwich plates with nonlinear elastic core using the small parameter method, *Vsnik Sxdnoukrans'kogo Naconal'nogo Unversitetu Men Volodimira Dalya* **165**(11):32-40.

[4] Magnucka-Blandzi E., Wittenbeck L., 2013, Approximate solutions of equilibrium equations of sandwich circular plate, *AIP Conference Proceedings* **1558**: 2352-2355.

[5] Magnucki K.A., Jasion P.A., Magnucka-Blandzi E.B., Wasilewicz P.A., 2014, Theoretical and experimental study of a sandwich circular plate under pure bending, *Thin-Walled Structures* **79**:1-7.

[6] Mixajlov I.P., 1969, Some problems of axisymmetric bending circular sandwich plates with rigid core, *Trudy Leningradskogo Korablestroitel'Nogo Instituta* **66**: 125-131.

[7] Noor A.K., Scott Burton W., Bert Ch. W., 1996, Computational models for sandwich panels and shells, *Applied Mechanics Reviews* **49**(3): 155-199.

[8] Prusakov A.P., 1961, Some problems bending circular sandwich plates with lightweight filler, *Tr. konf. po teor. plastin i obolochek* **1**:293-297.

[9] Ren-huai L., 1981, Nonlinear bending of circular sandwich plates, *Applied Mathematics and Mechanics* **2**(2): 189-208.

[10] Tamurov Yu.N., 1991, Vibrational processes in a three-layer shell with nonuniform compression of a physically nonlinear filler, *International Applied Mechanics* **27**(7): 698-703.

[11] Vinson J., Sierakowski R.,2002, The Behaviour of Streuctures Composed of Composite Materials, Martinus Nijhoff Publishers , Dordrecht , The Netherlands.

[12] Kauderer G., 1961, *Nonlinear Mechanics*, Russian Translation Available.

[13] Riahi A., Curran J.H., 2009, Full 3D finite element Cosserat formulation with application in layered structures, *Applied Mathematical Modelling* **33**:3450-3464.

[14] MacLaughlin M.M., Doolin D.M., 2006, Review of validation of the discontinuous deformation analysis (DDA) method, *International Journal for Numerical and Analytical Methods in Geomechanics* **30**(4):271-305.

[15] Pande G.N., Beer G., Williams J.R., 1990, *Numerical Methods in Rock Mechanics*, Wiley, Chichester.

[16] Williams J.R., O'Connor R., 1999, Discrete element simulation and the contact problem, *Archives of Computational Methods in Engineering* **6**(4): 279-304.

[17] Ambartsumian S., 1970, *Theory of Anisotropic Plates*, Technomic Stanford Fizmargiz, Moskva.

[18] Timoshenko S.P., Woinowsky-Krieger S., 1959,*Theory of Plates and Shells*, McGraw-Hill, New York.

[19] Hinch E.J., 1995, *Perturbation Methods*, Cambridge University Press.