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Analysis of Plane Waves in Anisotropic Magneto-Piezothermoelastic Diffusive Body with Fractional Order Derivative

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ABSTRACT

In this paper the propagation of harmonic plane waves in a homogeneous anisotropic magneto-piezothermoelastic diffusive body with fractional order derivative is studied. The governing equations for a homogeneous transversely isotropic body in the context of the theory of thermoelasticity with diffusion given by Sherief et al. [1] are considered as a special case. It is found that three types of waves propagate in one dimension anisotropic magneto-piezothermoelastic diffusive body, namely quasi-longitudinal wave (QP), quasi-thermal wave (QT) and quasi-diffusion wave (QD). The different characteristics of waves like phase velocity, attenuation coefficient, specific heat loss and penetration depth are computed numerically and presented graphically for Cadmium Selenide (CdSe) material. The effect of fractional order parameter on phase velocity, attenuation coefficient, specific heat loss and penetration depth has been studied.

Keywords : Piezothermoelastic; Magneto; Harmonic plane wave; Phase velocity; Attenuation coefficient.

1 INTRODUCTION

IN the recent years it has been seen an ever-growing interest in the investigation of models of an elastic body that take into account the influence of various physical fields such as thermal, electric, magnetic and other fields. An impetus for such studies was the creation of many new materials possessing properties that are not characteristic of usual elastic bodies. Among these materials are piezoelectric bodies that form the core of modern structures and instruments. A stressed state of a piezoelectric body is produced mainly by its deformation, as well as by thermal, magnetic and electric fields present in the body. Therefore a mathematical model magneto-piezothermoelastic quite adequately reflects the properties of such bodies.

The theory of thermopiezoelectric material was first proposed by Mindlin [3] and derived governing equations of a thermopiezoelectric plate. The physical laws for the thermopiezoelectric material have been explored by Nowacki [4,5]. Chandrasekharaiah [6] used generalised Mindlin's theory of thermopiezoelectricity to account for the finite speed of propagation of thermal disturbances.

Sharma [7] discussed the propagation of inhomogeneous waves in anisotropic piezothermoelastic media. Sharma & Kumar [8] discussed the plane harmonic waves in piezothermoelastic material. Sharma & Walia [9] investigated Rayleigh waves in transversely isotropic piezothermoelastic materials. Sharma et al. [10] studied the propagation characteristics of Rayleigh waves in transversely isotropic piezothermoelastic materials. Fatimah [11] presented the

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mathematical model for studying the influence of the initial stresses who relaxation waves in piezothermoelastic half-space.

Sherief et al. [1] developed the generalized theory of thermoelastic diffusion with one relaxation time, which allows finite speeds of propagation of waves. Singh [12,13] discussed the reflection phenomenon of waves from free surface of an elastic solid with generalized thermodiffusion. Aouadi [14–18] investigated different types of problems in thermoelastic diffusion. Sharma [19-21] discussed plane harmonic generalized thermoelastic diffusive waves and elasto-thermodiffusive surface waves in heat-conducting solids. Kumar and Kansal [22] analysed the plane wave propagation in an anisotropic thermoelastic diffusive body.

With the development of active material systems, there is significant interest in coupling effects between elastic, electric, magnetic and thermal fields, for their applications in sensing and actuation. Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. Van Run et al. [23] reported the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite which had the magnetoelectric effect not existing in either the constituent. Li and Dunn [24] quantitatively explained the magnetoelectric coupling created through the interaction between piezoelectric and piezomagnetic phases. Oatao and Ishihara [25] analysed the laminated hollow cylinder constructed of isotropic elastic and magneto-electro-thermoelastic material. Pang and Li [26] studied the SH interfacial waves between piezoelectric/piezomagnetic half-spaces with magneto-electro-elastic imperfect bonding. The effects of piezoelectric and piezomagnetic on the surface wave velocity of magneto-electro-elastic solids are studied by Li and Wei [27]. Abd-alla, Alshaikh, Giorgio, and Corte [28] studied the influence of the initial stress on propagation of longitudinal waves in a hollow infinite circular cylinder in the presence of an axial initial magnetic field.

Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Studied over the intervening three hundred years have proven at least half right. It is clear, that within the 20th century, especially numerous applications have been found. However these applications and mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult to grasp, the definitions are no more rigorous than integer order counterpart. Kumar and Gupta [29] studied the plane wave propagation in an anisotropic thermoelastic body with fractional order derivative and void. Bassiory and Sabry [30] discussed fractional order two temperature thermo-elastic behaviour of piezoelectric materials. Attenuated fractional wave equations in anisotropic media are studied by Meerschaert and McGough [31]. Kumar and Gupta [32] analysed the plane wave propagation and domain of influence in fractional order thermoelastic materials with three phase lag heat transfer. Meral and Royston [33] investigated the response of the fractional order on viscoelastic half space to surface and subsurface sources. Meral et al. [34] discussed the Rayleigh-Lamb wave propagation on a fractional order viscoelastic plate.

In this article, propagation of plane waves in an anisotropic magneto-piezothermoelastic diffusive body with fractional order derivative in one dimensional model has been investigated. The phase velocity and attenuation coefficient, specific heat loss and penetration depth of plane waves has been computed and presented graphically for different values of frequency. The analytical results have also been computed numerically and represented graphically for illustration of various physical phenomena occurring in such solids.

2 BASIC EQUATIONS

Following Sherief [1,2], Li [35], and Kuang [36], the basic equations for a homogeneous anisotropic magneto-piezothermoelastic diffusive body with fractional order derivative in the absence of body forces, free charge density, heat and mass diffusive sources are:

Constitutive equations:

$$\begin{gathered}
\sigma_{ij}=c_{ijkl}\,\varepsilon_{kl}-e_{ijk}E_k-\alpha_{ij}T-q_{ijk}H_k-b_{ij}C,\qquad -q_{i,i}=\rho T_0\dot{S},\qquad \rho S=\alpha_{ij}\varepsilon_{ij}+\tau_i E_i+rT+aC+m_i H_i,\\
-\eta_{i,i}=\dot{C},\qquad \mu=bC-b_{ij}\varepsilon_{ij}-b_i D_i+aS-d_i B_i,\qquad D_i=A_{ij}E_j+e_{ijk}\varepsilon_{jk}+\tau_i T+f_{ij}H_j+b_i C,\\
B_i=f_{ij}E_j+q_{ijk}\varepsilon_{jk}+\beta_{ij}H_j+m_i T+d_i C,\qquad E_i=-\phi_{,i},\qquad H_i=-\psi_{,i},\qquad (i,j,k,l=1,2,3)
\end{gathered} \tag{1}$$

Equations of motion:

$$\sigma_{ij,j}-\rho\ddot{u}_i=0, \tag{2}$$

Gauss equations:

$$D_{i,i}=0, \tag{3}$$

$$B_{i,i}=0, \tag{4}$$

Equation of heat conduction:

$$K_{ij}\,T_{,ij}=\left(1+\tau_0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)T_0\left(\alpha_{ij}\dot{u}_{i,j}-\tau_i\dot{\phi}_{,i}+r\dot{T}-m_i\dot{\psi}_{,i}+a\dot{C}\right), \tag{5}$$

Equation of chemical potential:

$$\alpha_{ij}^{\;*}\left(-b_{ij}u_{i,jji}+b_i\phi_{,iji}+aT_{,ji}+d_i\psi_{,iji}+bC_{,ji}\right)=\left(1+\tau^0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\dot{C}, \tag{6}$$

where c_{ijkl} are elastic parameters, $\alpha_{ij},\beta_{ij},f_{ij},m_i,q_{ijk},b_i,d_i,b_{ij},\alpha_{ij}^{\;*}$ are tensors of magneto-piezothermal and diffusion moduli respectively. ρ, C_e are, respectively , the density and specific heat at constant strain. *a,r,b* are, respectively, coefficients describing the measure of thermal and mass diffusion effects, q_i and η_i are the components of heat and mass diffusion flux vectors q and η respectively, S,μ are entropy and chemical potential per unit mass respectively, A_{ij},e_{ijk},τ_i are the piezoelectric coefficients , C is the mass concentration of the diffusion material in the elastic body, T is the absolute temperature of the body, T_0 is the reference temperature, τ_0 is the thermal relaxation time, and τ^0 is the diffusion relaxation time, which will ensure that the heat conduction equation will predict finite speeds of heat propagation of matter from one body to other. u_i are the components of displacement vector u, $\sigma_{ij}(=\sigma_{ji})$ are the components of the stress tensor, $\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ are the components of the strain tensor, $K_{ij}(=K_{ji})$ are the components of thermal conductivity, E_i is the electric field intensity, D_i is the electric displacement, H_i is the magnetic field intensity, B_i is the magnetic displacement, α is the fractional order such that $0<\alpha\le 1, \phi$ and ψ , are the electric and magnetic potentials. The symbols "," and "." corresponds to partial and time derivatives, respectively.

3 FORMULATION AND SOLUTION OF THE PROBLEM

We consider a homogeneous anisotropic magneto-piezothermoelastic diffusive body with fractional order derivative initially at the uniform temperature T_0. The governing equations in magneto-piezothermoelastic diffusive body with fractional order derivative are

$$\begin{gathered}
c_{ijkl}u_{k,lj}+e_{ijk}\phi_{,kj}+q_{ijk}\psi_{,kj}-\alpha_{ij}T_{,j}-b_{ij}C_{,j}-\rho\ddot{u}_i=0, \qquad e_{ijk}u_{j,ki}-A_{ij}\phi_{,ji}-f_{ij}\psi_{,ji}+\tau_iT_{,i}+b_iC_{,i}=0,\\
q_{ijk}u_{j,ki}-f_{ij}\phi_{,ji}-\beta_{ij}\psi_{,ji}+m_iT_{,i}+d_iC_{,i}=0, \qquad K_{ij}\,T_{,ij}=\left(1+\tau_0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)T_0\left(\alpha_{ij}\dot{u}_{i,j}-\tau_i\dot{\phi}_{,i}+r\dot{T}-m_i\dot{\psi}_{,i}+a\dot{C}\right),\\
\alpha_{ij}^{\;*}\left(-b_{ij}u_{i,jji}+b_i\phi_{,iji}+aT_{,ji}+d_i\psi_{,iji}+bC_{,ji}\right)=\left(1+\tau^0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\dot{C}, \quad (i,j,k=1,2,3)
\end{gathered} \tag{7}$$

For plane harmonic waves, we assume the wave solution as:

$$\left(u_k,\phi,\psi,T,C\right)=\left(\bar{u}_k,\bar{\phi},\bar{\psi},\bar{T},\bar{C}\right)\exp\left[i\left(\xi n_k x_k-\omega t\right)\right] \quad , \ k=1,2,3 \tag{8}$$

where ω is the angular frequency and ξ is the complex wave number. $\bar{u}_k,\bar{\phi},\bar{\psi},\bar{T}$ and $\bar{C}$ are the undetermined amplitude vectors that are independent of time t and coordinates x_1,x_2 and x_3. Upon using Eq. (8) in system of Eqs. (7) with the aid of system of Eqs. (1), yields

$$\begin{aligned}
&\left(c_{ijkl}n_l n_j\xi^2-\rho\omega^2\right)\bar{u}_k+e_{ijk}n_k n_j\xi^2\bar{\phi}+q_{ijk}n_k n_j\xi^2\bar{\psi}+i\alpha_{ij}n_j\xi\bar{T}+ib_{ij}n_j\xi\bar{C}=0,\\
&-e_{ijk}n_k n_i\xi^2\bar{u}_j+A_{ij}n_i n_j\xi^2\bar{\phi}+f_{ij}n_i n_j\xi^2\bar{\psi}+i\tau_i n_i\xi\bar{T}+ib_i n_i\xi\bar{C}=0,\\
&-q_{ijk}n_k n_i\xi^2\bar{u}_j+f_{ij}n_i n_j\xi^2\bar{\phi}+\beta_{ij}n_i n_j\xi^2\bar{\psi}+im_i n_i\xi\bar{T}+id_i n_i\xi\bar{C}=0,\\
&i\tau_1\alpha_{ij}n_j\xi\bar{u}_i-i\tau_1\tau_i n_i\xi\bar{\phi}-i\tau_1 m_i n_i\xi\bar{\psi}+\left(\tau_1 r-K_{ij}n_i n_j\xi^2\right)\bar{T}+\tau_1 a\bar{C}=0,\\
&i\alpha_{ij}{}^{*}n_i n_j b_{ij}n_j\xi^3\bar{u}_i-i\alpha_{ij}{}^{*}n_i n_j b_i n_i\xi^3\bar{\phi}-i\alpha_{ij}{}^{*}n_i n_j d_i n_i\xi^3\bar{\psi}-\alpha_{ij}{}^{*}n_i n_j a\xi^2\bar{T}+\left(\tau_2-\alpha_{ij}{}^{*}n_i n_j b\xi^2\right)\bar{C}=0, \qquad (i,j,k,l=1,2,3)
\end{aligned}$$

where

$$\tau_1=i\omega\left(1-\tau_0(i\omega)^{\alpha+1}\right)T_0\ ,\ \tau_2=i\omega\left(1-\tau^0(i\omega)^{\alpha+1}\right)$$

We introduce the Christoffel's notation as follows:

$$\begin{aligned}
&\gamma_{ij}=c_{ijkl}n_k n_j\ ,\ e_i=e_{ijk}n_j n_k\ ,\ \alpha_i=\alpha_{ij}n_j\ ,\ f=f_{ij}n_i n_j\ ,\ \beta=\beta_{ij}n_i n_j\ ,\ A=A_{ij}n_i n_j\ ,\ \tau^*=\tau_i n_i\ ,\\
&q_i=q_{ijk}n_j n_k\ ,\ \ m=m_i n_i\ ,\ K_1=K_{ij}n_i n_j\ ,\ d=d_i n_i\ ,\ b^*=b_i n_i\ ,\ \alpha^*=\alpha_{ij}{}^{*}n_i n_j\ ,\ b_i=b_{ij}n_j,
\end{aligned}$$

Then field equations in a homogeneous anisotropic magneto-piezothermoelastic diffusive body with fractional order derivative in one dimension are

$$\begin{aligned}
&\left(\gamma_{11}\xi^2-\rho\omega^2\right)\bar{u}_1+e_1\xi^2\bar{\phi}+q_1\xi^2\bar{\psi}+i\alpha_1\xi\bar{T}+ib_1\xi\bar{C}=0,\\
&-e_1\xi^2\bar{u}_1+A\xi^2\bar{\phi}+f\xi^2\bar{\psi}+i\tau^*\xi\bar{T}+ib^*\xi\bar{C}=0,\\
&-q_1\xi^2\bar{u}_1+f\xi^2\bar{\phi}+\beta\xi^2\bar{\psi}+im\xi\bar{T}+id\xi\bar{C}=0,\\
&i\tau_1\alpha_1\xi\bar{u}_1-i\tau_1\tau^*\xi\bar{\phi}-i\tau_1 m\xi\bar{\psi}+\left(\tau_1 r-K_1\xi^2\right)\bar{T}+\tau_1 a\bar{C}=0,\\
&i\alpha^* b_1\xi^3\bar{u}_i-i\alpha^* b^*\xi^3\bar{\phi}-i\alpha^* d_1\xi^3\bar{\psi}-\alpha^* a\xi^2\bar{T}+\left(\tau_2-\alpha^* b\xi^2\right)\bar{C}=0,
\end{aligned} \tag{9}$$

Eq. (9) represents a linear system of five homogeneous equations in five unknowns $\bar{u}_1,\bar{\phi},\bar{\psi},\bar{T}$ and $\bar{C}$ which possesses non- trivial solution if the determinant of the coefficients $\left[\bar{u}_1,\bar{\phi},\bar{\psi},\bar{T},\bar{C}\right]^{tr}$ vanishes i.e.

$$\begin{vmatrix}
\left(\gamma_{11}\xi^2-\rho\omega^2\right) & e_1\xi^2 & q_1\xi^2 & i\alpha_1\xi & ib_1\xi\\
-e_1\xi^2 & A\xi^2 & f\xi^2 & i\tau^*\xi & ib^*\xi\\
-q_1\xi^2 & f\xi^2 & \beta\xi^2 & im\xi & id\xi\\
i\tau_1\alpha_1\xi & -i\tau_1\tau^*\xi & -i\tau_1 m\xi & \left(\tau_1 r-K_1\xi^2\right) & \tau_1 a\\
i\alpha^* b_1\xi^3 & -i\alpha^* b^*\xi^3 & -i\alpha^* d\xi^3 & -\alpha^* a\xi^2 & \left(\tau_2-\alpha^* b\xi^2\right)
\end{vmatrix}=0 \tag{10}$$

The Eq. (10) yields to the following polynomial characteristic equation in ξ as:

$$p_{11}\xi^6 + p_{12}\xi^4 + p_{13}\xi^2 + p_{14} = 0 \ , \ \ \xi^4 = 0. \tag{11}$$

The coefficients $p_{11}, p_{12}, p_{13}, p_{14}$ are given in the appendix A. Solving Eq. (11) , we obtain three roots of ξ, in which we are interested to those roots whose imaginary parts are positive because only those roots give the negative roots of the decay coefficient $\mathrm{Im}(\xi)$. Corresponding to these roots, there exist three waves corresponding to descending order of their velocities, namely quasi-longitudinal wave (QP), quasi-thermal wave (QT) and quasi-diffusion wave (QD). We denote the values of ξ associated with these modes by ξ_1, ξ_2 and ξ_3 respectively. The offer phase velocity, attenuation coefficient, specific heat loss and penetration depth of these types of waves:

(i) Phase velocity: The phase velocity is given by

$$V_i = \frac{\omega}{\left|\mathrm{Re}(\xi_i)\right|}, \quad i = 1,2,3 \tag{12}$$

where V_1, V_2 and V_3 are the velocities of the quasi-longitudinal wave (QP), quasi-thermal wave (QT) and quasi-diffusion wave (QD) modes respectively.

(ii) Attenuation coefficient: The attenuation coefficient is defined as:

$$Q_i = \mathrm{Im}(\xi_i), \quad i = 1,2,3 \tag{13}$$

where Q_1, Q_2 and Q_3 are the attenuation coefficients of the quasi-longitudinal wave (QP), quasi-thermal wave (QT) and quasi-diffusion wave (QD) modes respectively.

(iii) Specific heat loss: The specific heat loss is given by

$$SP_i = 4\pi \left|\frac{\mathrm{Img}(\xi_i)}{\mathrm{Re}(\xi_i)}\right|, \quad i = 1,2,3 \tag{14}$$

where SP_1, SP_2 and SP_3 are the specific heat loss of the quasi-longitudinal wave (QP), quasi-thermal wave (QT) and quasi-diffusion wave (QD) modes respectively.

(iv) Penetration depth: The penetration depth is defined as:

$$PD_i = \frac{1}{\left|\mathrm{Im}(\xi_i)\right|}, \quad i = 1,2,3 \tag{15}$$

where PD_1, PD_2 and PD_3 are the penetration depth of the quasi-longitudinal wave (QP), quasi-thermal wave (QT) and quasi-diffusion wave (QD) modes respectively.

4 TRANSVERSELY ISOTROPIC MEDIA

Following Slaughter [37], applying transformation in Eqs. (2)- (6), the basic governing equations for a homogeneous transversely isotropic, magneto-piezothermoelastic diffusive body with fractional order derivative in one dimension can be written as:

$$\begin{aligned}
&c_{11}u_{1,11}+e_{11}\phi_{,11}+q_{11}\psi_{,11}-\alpha_1 T_{,1}-b_1 C_{,1}=\rho\ddot{u}_1,\\
&e_{11}u_{1,11}-A_{11}\phi_{,11}-f_{11}\psi_{,11}+\tau_1 T_{,1}+b_1^{*}C_{,1}=0,\\
&q_{11}u_{1,11}-f_{11}\phi_{,11}-\beta_{11}\psi_{,11}+m_1 T_{,1}+d_{,1}C_{,1}=0,\\
&K_1 T_{,11}=\left(1+\tau_0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)T_0\left(\alpha_1\dot{u}_{1,1}-\tau_1\dot{\phi}_{,1}-m_1\dot{\psi}_{,1}+r\dot{T}+a\dot{C}\right),\\
&\alpha^{*}\left(-b_1 u_{1,111}+b_1^{*}\phi_{1,111}+d_1\psi_{1,111}+aT_{,11}+bC_{,11}\right)=\left(1+\tau^0\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\dot{C},
\end{aligned} \tag{16}$$

where

$$\alpha_1=c_{11}\alpha_t\ ,\ b_1=c_{11}\alpha_c\,,$$

Here, α_t,α_c are the coefficients of thermal and diffusion expansion. In Eqs. (16) we have used the contracting subscript notations $1\to 11$ to relate $c_{1111}\to c_{11}$ and so on.

Using Eq. (8) in Eqs. (16), we obtain the following characteristics equation

$$p_{11}{}^{*}\xi^6+p_{12}{}^{*}\xi^4+p_{13}{}^{*}\xi^2+p_{14}{}^{*}=0\quad,\quad \xi^4=0. \tag{17}$$

The coefficients $p_{11}{}^{*},p_{12}{}^{*},p_{13}{}^{*},p_{14}{}^{*}$ are given in appendix B. In the coefficients $p_{1i}{}^{*},\ i=1,2,3,4$, we have used the following dimensionless quantities:

$$\left(x_1{}',u_1{}'\right)=\frac{\omega_1}{c_1}\left(x_1,u_1\right),\ \left(t',\tau_0{}',\tau^0{}'\right)=\omega_1\left(t,\tau_0,\tau^0\right),\ T'=\frac{\alpha_1}{\rho c_1^2}T\ ,\ \phi'=\frac{\omega_1 e_{11}\phi}{c_1\alpha_1 T_0},\psi'=\frac{\omega_1 q_{11}\psi}{c_1\alpha_1 T_0} \tag{18}$$

where

$$c_1=\sqrt{\frac{c_{11}}{\rho}} \tag{19}$$

Here, $\omega_1=\dfrac{\rho C_e c_1^2}{K_1}$ is the characteristic frequency of the body, $c_1=\sqrt{\dfrac{c_{11}}{\rho}}$ is the longitudinal wave velocity in the body.

If $\alpha=1$, we obtain the characteristic polynomial equation which is similar to that if we solve the problem directly without fractional order derivative.

If we neglect the magnetic effect i.e. $q_1=0,f=0,\beta=0,m=0,d=0$, then we obtain the characteristic polynomial equation for a piezothermoelastic diffusive body with fractional order.

If we neglect the piezoelectric effect i.e. $e_1=0,f=0,A=0,\tau^{*}=0,b^{*}=0$, then we obtain the characteristic polynomial equation for a magneto-thermoelastic diffusive body with fractional order.

5 NUMERICAL RESULTS AND DISCUSSIONS

For the purpose of numerical calculation, we consider the case of an anisotropic media. We can solve Eq. (17) with the help of the software Matlab 7.8 & using the formulas given by Eqs. (18), (19), we can compute the phase velocity, attenuation coefficient, specific heat loss and penetration depth for intermediate values of angular frequency (ω). Following Vashishth and Sukhija [13] and Kumar and Kansal [22], the numerical values have been taken as follows:

$$\gamma_{11} = 7.41\times10^{10}\, Nm^{-2}, \rho = 5500\, Kgm^{-3},\ m = 2.1\times10^{-2}\ KgC^{-1}K^{-1}s^{-1},\ \alpha^* = 1.05\times10^{-8} Kgm^{-3}s$$
$$\alpha_1 = 0.621\times10^6\ NK^{-1}m^{-2},\ T_0 = 298K, \tau_0 = 0.02s,\ q_1 = 4.5\times10^{-2} NA^{-1}m^{-1}, b^* = 4.7\times10^{-2} Kg^{-1}Cm,$$
$$\tau^* = -2.94\times10^{-6}\ CK^{-1}m^{-2}, a = 8.2\times10^{-5}\ K^{-1}m^2s^{-2}, \beta = 6.7\times10^{-3}\ NC^{-2}s^2, b = 2.3\times10^{-2}\, m^5Kg^{-1}s^{-2},$$
$$f = 5.8\times10^{-2} N\ V^{-1}C^{-1}s,\ A = 8.26\times10^{-11}\ Fm^{-1},\ C_e = 260\, JKg^{-1}K^{-1},\ K_1 = 9\, WK^{-1}m^{-1},$$
$$b_1 = 2.14\times10^{-2} NKg^{-1}m, e_1 = 5\times10^{-2} Cm^{-2},\ \tau_0 = 0.02s,\ \tau^0 = 0.03s,\ d = 5.3\times10^{-2} A^{-1}ms^{-2}.$$

In all the graphs, notations ▬ ALP 0.25, ‧‧‧‧ ALP0.75, ‧ ‧ ‧ ALP1 denote the curves of phase velocities, attenuation coefficients, specific heat loss and penetration depth of waves corresponding to the different values of fractional order parameter i.e. $\alpha = 0.25, 0.75$ and $\alpha = 1$, respectively.

Figs. 1, 2,3 show the variations of the phase velocities V_1, V_2 and V_3 of waves with respect to ω. Figs. 4,5,6 show the variations of the attenuation coefficients Q_1, Q_2 and Q_3 of waves respectively. Figs. 7, 8, 9 show the variations of the specific heat loss SP_1, SP_2 and SP_3 of waves respectively. Figs. 10,11,12 show the variations of the penetration depth PD_1, PD_2 and PD_3 of waves respectively.

5.1 Phase velocity

It is clear from Fig. 1, that phase velocity V_1 of QP wave monotonically increases and then tends to decrease with increase in angular frequency (ω) for different values of α i.e. $\alpha = 0.25, 0.75, 1$. But for $\alpha = 0.25, V_1$, is maximum i.e. least value of α corresponds to the highest value of phase velocity.

For $1 \le \omega \le 4$, V_2 increases strictly and shows a quick downfall. Among the different values of α, V_2, possesses highest magnitude value for $\alpha = 0.25$.

Fig. 3 shows that for $1 \le \omega \le 4$, V_3 increases then it shows a sudden change in slope but behaviour remains the same.

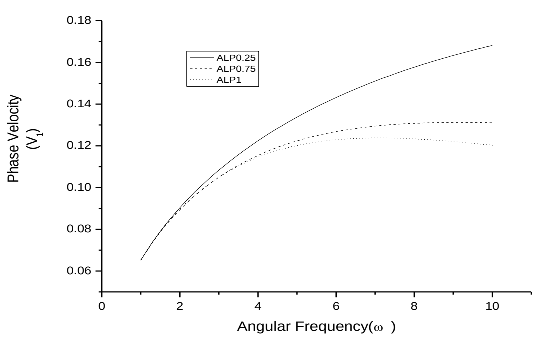

Fig.1
Variation of phase velocity w.r.t angular frequency (quasi-longitudinal wave).

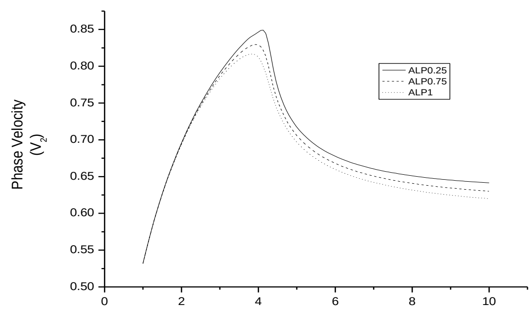

Fig.2
Variation of phase velocity w.r.t angular frequency (quasi-thermal wave).

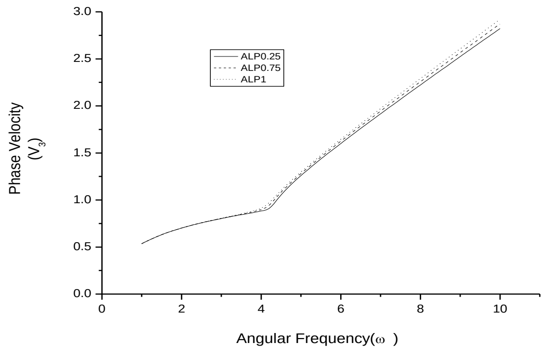

Fig.3
Variation of phase velocity w.r.t angular frequency (quasi-diffusion wave).

5.2 Attenuation coefficient

It is clear from Fig. 4 that the attenuation coefficient Q_1 of quasi-longitudinal (QP) wave strictly increases with difference in magnitude values for different values of α with increase in angular frequency. For $\alpha = 0.25$, it possesses least magnitude value.

Fig. 5, 6 show the behaviour of Q_2 and Q_3 opposite to each other. From Fig. 5, it is clear that Q_2 for $2 \le \omega \le 4 \cdot 5$ shows a downfall in values and then tends to increase so that it gains highest value for $\alpha = 1$. Fig. 6 shows the trend of Q_3 just revert to Q_2.

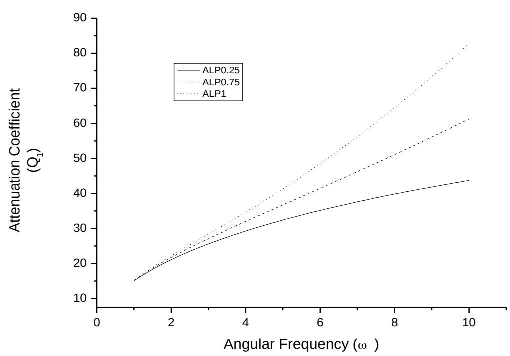

Fig.4
Variation of attenuation coefficient w.r.t angular frequency (quasi-longitudinal wave).

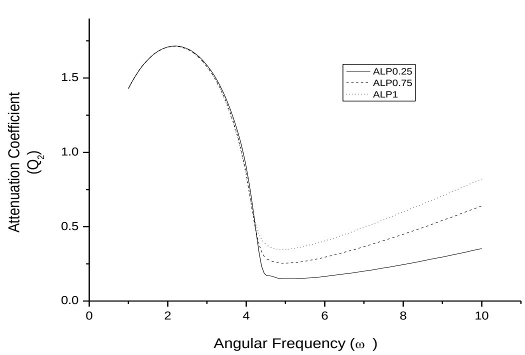

Fig.5
Variation of attenuation coefficient w.r.t angular frequency (quasi-thermal) wave).

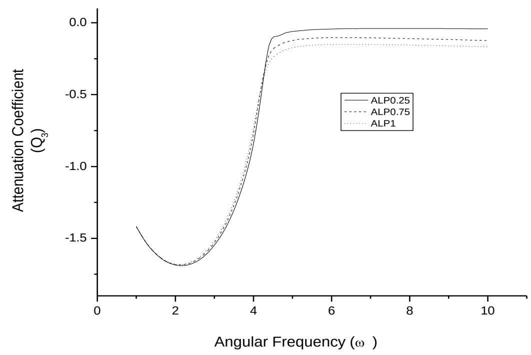

Fig.6
Variation of attenuation coefficient w.r.t angular frequency (quasi-diffusion wave).

5.3 Specific heat loss

Fig. 7 depicts that the specific heat loss SP_1 of quasi-longitudinal wave decreases more with decrease in α and increase in angular frequency.

It is clear from Fig. 8 that SP_2 shows a decreasing trend for $4 \le \omega \le 4 \cdot 5$ and then tends to increase as angular frequency increases and it is least for least value of α .

Fig. 9 shows that initially decreases for $1 \le \omega \le 4 \cdot 5$ and possesses similar magnitudes values for different values of α , following a stationary behaviour as angular frequency increases.

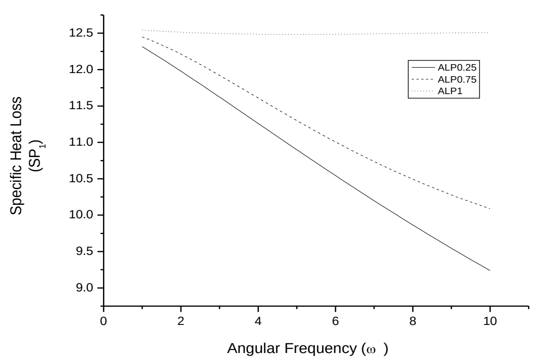

Fig.7
Variation of specific heat w.r.t angular frequency (quasi-longitudinal wave).

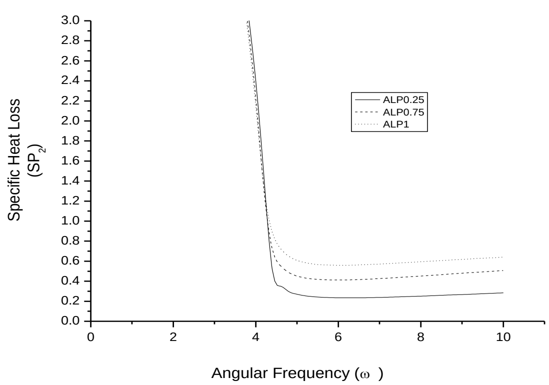

Fig.8
Variation of specific heat w.r.t angular frequency (quasi-thermal wave).

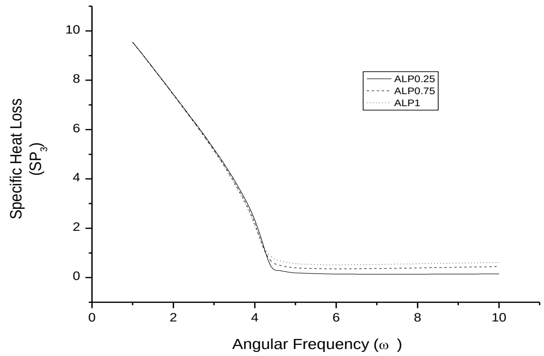

Fig.9
Variation of specific heat w.r.t angular frequency (quasi-diffusion wave).

5.4 Penetration depth

We noticed that the Fig. 10 show that the penetration depth PD_1 of the quasi-longitudinal wave monotonically decreases with increase in ω . For least value of α PD_1 is maximum.

It is clear from Fig. 11 that the penetration depth PD_2 of quasi-thermal wave and its peak value is obtained $\omega = 5$ and further it decreases. For least value of α PD_2 is maximum.

For $1 \le \omega \le 4 \cdot 5$ penetration depth PD_3 of quasi diffusion wave is constant for all values of α and then tends to decrease as angular frequency increases but for $\alpha = 1$, it gains in its numerical value in comparison to the other values of fractional order derivative.

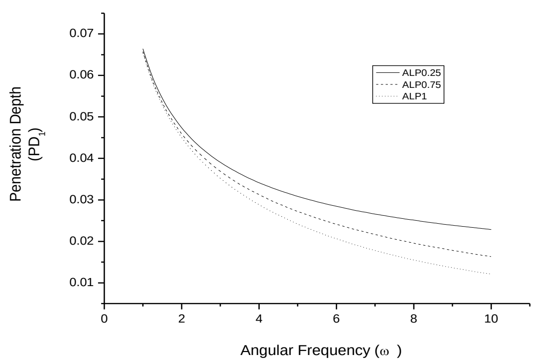

Fig.10
Variation of specific heat w.r.t angular frequency (quasi-longitudinal wave).

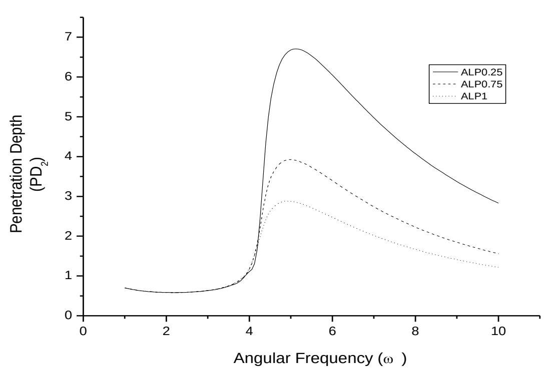

Fig.11
Variation of specific heat w.r.t angular frequency (quasi-thermal wave).

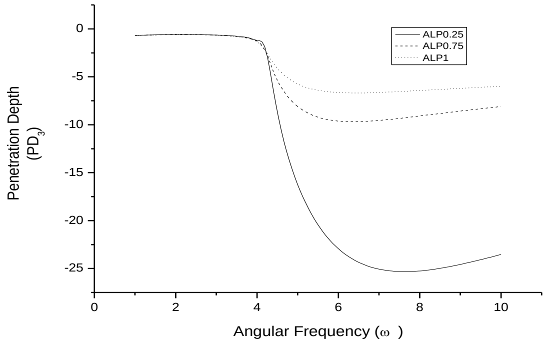

Fig.12
Variation of specific heat w.r.t angular frequency (quasi-diffusion wave).

6 CONCLUSIONS

Analysis of plane wave propagation is a significant problem of continuum mechanics. The propagation of plane harmonic waves in a homogeneous, anisotropic magneto-piezothermoelastic diffusive body with fractional order derivative has been studied. The anisotropic variations of phase velocities, attenuation coefficients, specific heat loss and penetration depth depending upon the fractional order derivative are observed in the context of theory of thermoelasticity with diffusion given by Sherief et al. [1,2]. All the field quantities are found to be sensitive towards the fractional order parameter.

1. A predominant effect of fractional order on quasi-longitudinal wave (QP), quasi-thermal wave (QT) and quasi-diffusion wave (QD is observed for the values of thermal and diffusion relaxation times.
2. It is observed that the phase velocities of QP and QT waves show an alternating behaviour but phase velocity of QD wave strictly increases. For $\alpha = 0.25$ the phase velocities V_1 and V_2 possess the highest magnitude value whereas the attenuation coefficients Q_1 and Q_2 least magnitude value in comparison to the other values of α .
3. For $\alpha = 0.25$ minimum specific heat loss is observed for all the waves.
4. Behaviour and trend of values of penetration depth of the waves is similar for all considered values of α with difference in their magnitude values.

APPENDIX A

$$p_{11} = g_{11} + g_{15} + g_{18} + g_{23},\ p_{12} = g_{12} + g_{19} + g_{21} + g_{24},\ p_{13} = g_{13} + g_{17} + g_{20} + g_{22},\ p_{14} = g_{14},$$
$$g_{11} = \gamma_{11} d_{11},\ g_{12} = \gamma_{11} d_{12} - a_{11} d_{11},\ g_{13} = \gamma_{11} d_{13} - a_{11} d_{12},\ g_{14} = -a_{11} d_{13},\ g_{15} = -e_1 d_{14},\ g_{16} = -e_1 d_{15},$$
$$g_{17} = -e_1 d_{16}, g_{18} = q_1 d_{17}, g_{19} = q_1 d_{18}, g_{20} = q_1 d_{19}, g_{21} = \alpha_1 d_{20}, g_{22} = \alpha_1 d_{21}, g_{23} = -b_1 d_{22}, g_{24} = -b_1 d_{23},$$
$$d_{11} = A\beta b_{11} - db_{15} A + f^2 b_{11} - fdb_{20} - \beta b^* b_{20} + b^* fb_{15},\ d_{12} = A\beta b_{12} - dAb_{14} + Amb_{16} + f^2 b_{12} + fmb_{18} \ldots$$
$$-fdb_{21} - \tau^* db_{22} + b^* mb_{22} + \tau^* \beta b_{18} - \tau^* fb_{18} + b^* fb_{14} - b^* \beta b_{21},\ d_{13} = \beta Ab_{13} - mAb_{17} - f^2 b_{13} \ldots$$
$$+mfb_{19} + \tau^* fb_{17} + \tau^* \beta b_{19},\ d_{14} = -e_1 \beta b_{11} + e_1 db_{15} + fq_1 b_{11} + fdb_{25} - b^* q_1 b_{15} - b^* \beta b_{25},$$
$$d_{15} = e_1 \beta b_{12} - e_1 mb_{16} + fq_1 b_{12} - fmb_{23} + fdb_{26} + \tau^* q_1 b_{16} + \tau^* \beta b_{23} - b^* \beta b_{26} - b^* mb_{27} + \tau^* b_{27} - b^* q_1 b_{14} \ldots$$
$$+e_1 db_{14}, d_{16} = -e_1 \beta b_{13} + e_1 mb_{17} + fq_1 b_{13} - fmb_{24} - \tau^* q_1 b_{17} + \tau^* \beta b_{24}, d_{17} = -e_1 fb_{11} + e_1 db_{15} + Aq_1 b_{11} \ldots$$
$$+Adb_{25} - b^* q_1 b_{15} - b^* fb_{25},\ d_{18} = -e_1 fb_{12} - e_1 mb_{28} + Aq_1 b_{12} - Amb_{23} + Adb_{26} + \tau^* q_1 b_{28} + \tau^* fb_{23} - b^* fb_{26} \ldots$$
$$-b^* mb_{30} + \tau^* db_{30} - b^* q_1 b_{29} + e_1 db_{29},\ d_{19} = -e_1 fb_{13} + e_1 mb_{19} + Aq_1 b_{13} - Amb_{24} - \tau^* q_1 b_{19} + \tau^* fb_{24},$$
$$d_{20} = -e_1 fb_{16} - e_1 \beta b_{28} - e_1 db_{31} + Aq_1 b_{16} + A\beta b_{23} + Adb_{27} - fq_1 b_{28} - f^2 b_{23} - dfb_{30} - b^* q_1 b_{30} + b^* fb_{27} + \ldots$$
$$b^* \beta b_{30},\ d_{21} = e_1 fb_{17} - e_1 \beta b_{19} - Aq_1 b_{17} + A\beta b_{24} + fq_1 b_{19} - f^2 b_{24}, d_{22} = -e_1 fb_{15} - e_1 \beta b_{20} + Aq_1 b_{15} + A\beta b_{25} \ldots$$
$$-fq_1 b_{20} - f^2 b_{25},\ d_{23} = -e_1 fb_{14} + e_1 \beta b_{29} - e_1 mb_{31} + Aq_1 b_{14} + A\beta b_{32} + Amb_{27} - fq_1 b_{29} - f^2 b_{32} - mfb_{30} \ldots$$
$$+\tau^* q_1 b_{31} - \tau^* fb_{27} + \tau^* \beta b_{30},$$

$b_{11}=K_1a_{21},\ b_{12}=-a_{15}a_{21}-K_1\tau_2+a_{16}a_{20},\ b_{13}=a_{15}\tau_2,\ b_{14}=a_{14}a_{20}+a_{15}a_{19},\ b_{15}=K_1a_{19},\ b_{16}=a_{14}a_{21}+a_{16}a_{19},$
$b_{17}=a_{14}\tau_2,\ b_{18}=a_{13}a_{21}-a_{16}a_{18},\ b_{19}=a_{13}\tau_2,\ b_{20}=K_1a_{18}, b_{21}=a_{13}a_{20}-a_{15}a_{18},\ b_{22}=-a_{13}a_{19}-a_{14}a_{18},$
$b_{23}=a_{12}a_{21}-a_{16}a_{17},\ b_{24}=a_{12}\tau_2,\ b_{25}=K_1a_{17}, b_{26}=-a_{15}a_{17}-a_{12}a_{20},\ b_{27}=a_{14}a_{17}-a_{17}a_{19}, b_{28}=a_{13}a_{21}+a_{16}a_{18},$
$b_{29}=a_{13}a_{20}+a_{15}a_{18}, b_{30}=-a_{12}a_{18}+a_{13}a_{17}, b_{31}=-a_{13}a_{19}+a_{14}a_{18}, b_{32}=-a_{12}a_{20}-a_{15}a_{17},$
$a_{11}=\rho\omega^2,\ a_{12}=\tau_1\alpha_1,\ a_{13}=\tau_1\tau^*,\ a_{14}=\tau_1 m,\ a_{15}=\tau_1 r, a_{16}=\tau_1 a, a_{17}=b_1\alpha^*,\ a_{18}=b^*\alpha^*, a_{19}=d\alpha^*, a_{20}=a\alpha^*, a_{21}=b\alpha^*,$

where

$$\tau_1=i\omega(1-(i\omega)^{\alpha+1}\tau_0)T_0, \tau_2=i\omega(1-(i\omega)^{\alpha+1}\tau^0).$$

APPENDIX B

$p_{11}{}^*=g_{11}+g_{15}+g_{18}+g_{23},\ p_{12}{}^*=g_{12}+g_{19}+g_{21}+g_{24},\ p_{13}{}^*=g_{13}+g_{17}+g_{20}+g_{22},\ p_{14}{}^*=g_{14},$
$g_{11}=-d_{11},\ g_{12}=\omega^2d_{12}-a_{11}d_{11},\ g_{13}=-d_{13}+\omega^2d_{12},\ g_{14}=\omega^2d_{13},\ g_{15}=a_{11}d_{14},\ g_{16}=a_{11}d_{15},$
$g_{17}=a_{11}d_{16}, g_{18}=-a_{11}d_{17}, g_{19}=-a_{11}d_{18}, g_{20}=-a_{11}d_{19}, g_{21}=-d_{20}, g_{22}=-d_{21}, g_{23}=d_{22}, g_{24}=d_{23},$
$d_{11}=a_{12}a_{16}b_{11}-a_{18}b_{15}a_{12}+a_{13}{}^2b_{11}-a_{13}a_{18}b_{20}-a_{16}a_{15}b_{20}+a_{15}a_{13}b_{15},\ d_{12}=a_{12}a_{16}b_{12}-a_{18}a_{12}b_{14}+a_{13}{}^2b_{12}+\ldots$
$a_{13}a_{17}b_{18}-a_{13}a_{18}b_{21}-a_{14}a_{18}b_{22}+a_{15}a_{17}b_{22}+a_{14}a_{16}b_{18}-a_{14}a_{13}b_{18}+a_{15}a_{13}b_{14}-a_{15}a_{16}b_{21},\ d_{13}=a_{16}a_{12}b_{13}\ldots$
$-a_{17}a_{12}b_{17}-a_{13}{}^2b_{13}+a_{17}a_{13}b_{19}+a_{14}a_{13}b_{17}+a_{14}a_{16}b_{19},\ d_{14}=-a_{16}b_{11}+a_{18}b_{15}+a_{13}b_{11}+a_{13}a_{18}b_{25}-a_{15}b_{15}\ldots$
$-a_{15}a_{16}b_{25}, d_{15}=-a_{16}b_{12}-a_{17}b_{16}+a_{13}b_{12}-a_{13}a_{17}b_{23}+a_{13}a_{18}b_{26}+a_{14}b_{16}+a_{14}a_{16}b_{23}-a_{15}a_{16}b_{26}-a_{15}a_{17}b_{27}\ldots$
$+a_{14}a_{18}b_{27}-a_{15}b_{14}+a_{18}b_{14}, d_{16}=-a_{16}b_{13}+a_{17}b_{17}+a_{13}b_{13}-a_{13}a_{17}b_{24}-a_{14}b_{17}+a_{14}a_{16}b_{24},$
$d_{17}=-a_{13}b_{11}+a_{18}b_{15}+a_{12}b_{11}+a_{12}a_{18}b_{25}-a_{15}b_{15}-a_{15}a_{13}b_{25},\ d_{18}=-a_{13}b_{12}-a_{17}b_{28}+a_{12}b_{12}-a_{12}a_{17}b_{23}\ldots$
$+a_{12}a_{18}b_{26}+a_{14}b_{28}+a_{14}a_{13}b_{23}-a_{15}a_{13}b_{26}-a_{15}a_{17}b_{30}+a_{14}a_{18}b_{30}-a_{15}b_{29}+a_{18}b_{29},$
$d_{19}=-a_{13}b_{13}+a_{17}b_{19}+a_{12}b_{13}-a_{12}a_{17}b_{24}-a_{14}b_{19}+a_{14}a_{13}b_{24}, d_{20}=-a_{13}b_{16}+a_{16}b_{28}-a_{18}b_{31}+a_{12}b_{16}+a_{12}a_{16}b_{23}\ldots$
$+a_{12}a_{18}b_{27}-a_{13}b_{28}-a_{13}{}^2b_{23}-a_{18}a_{13}b_{30}-a_{15}b_{30}+a_{15}a_{13}b_{27}+a_{15}a_{16}b_{30},$
$d_{21}=-a_{13}b_{17}-a_{16}b_{19}-a_{12}b_{16}+a_{12}a_{16}b_{24}+a_{13}b_{19}-a_{13}{}^2b_{24}, d_{22}=-e_1a_{13}b_{15}-e_1a_{16}b_{20}+a_{12}q_1b_{15}\ldots$
$+a_{12}a_{16}b_{25}-a_{13}q_1b_{20}-a_{13}{}^2b_{25},\ d_{23}=-e_1a_{13}b_{14}+e_1a_{16}b_{29}-e_1a_{17}b_{31}+a_{12}q_1b_{14}+a_{12}a_{16}b_{32}+a_{12}a_{17}b_{27}-a_{13}q_1b_{29}\ldots$
$-a_{13}{}^2b_{32}-a_{17}a_{13}b_{30}+a_{14}b_{31}-a_{14}a_{13}a_{27}+a_{14}a_{16}b_{30},$
$b_{11}=a_{38},\ b_{12}=a_{32}a_{38}-\tau_1^t-a_{33}a_{37},\ b_{13}=-a_{32}\tau_1^t,\ b_{14}=-a_{37}a_{31}-a_{32}a_{36},\ b_{15}=-a_{36},\ b_{16}=-a_{31}a_{38}-a_{33}a_{36},$
$b_{17}=-a_{38}\tau_1^t,\ b_{18}=-a_{30}a_{38}+a_{33}a_{35},\ b_{19}=-a_{30}\tau_1^t,\ b_{20}=a_{35}, b_{21}=-a_{30}a_{38}+a_{35}a_{33},\ b_{22}=a_{30}a_{36}+a_{31}a_{35},$
$b_{23}=a_{38}a_{21}-a_{33}a_{34},\ b_{24}=-a_{29}\tau_1^t,\ b_{25}=-a_{34}, b_{26}=a_{29}a_{37}-a_{32}a_{34},\ b_{27}=a_{29}a_{36}+a_{31}a_{34}, b_{28}=-a_{30}a_{38}-a_{33}a_{35},$
$b_{29}=-a_{30}\tau_1^t, b_{30}=-a_{29}a_{35}-a_{30}a_{34}, b_{31}=a_{30}a_{36}-a_{31}a_{35}, b_{32}=a_{12}a_{37}-a_{32}a_{17},$
$a_{29}=a_{19}\tau_0^t,\ a_{30}=a_{20}\tau_0^t, a_{31}=a_{21}\tau_0^t,\ \ a_{32}=a_{22}\tau_0^t,\ a_{33}=a_{23}\tau_0^t, a_{34}=-a_{24}\alpha_1{}^*, a_{35}=-a_{25}\alpha_1{}^*,$
$a_{36}=-a_{26}\alpha_1{}^*, a_{37}=-a_{27}\alpha_1{}^*, a_{38}=-a_{28}\alpha_1{}^*$

where

$$\tau_0^t=-i\omega(1-(i\omega)^{\alpha+1}\tau_0),\ \ \tau_1^t=-i\omega(1-(i\omega)^{\alpha+1}\tau^0).$$

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