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Axisymmetric Problem of Thick Circular Plate with Heat Sources in Modified Couple Stress Theory

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ABSTRACT

The main aim is to study the two dimensional axisymmetric problem of thick circular plate in modified couple stress theory with heat and mass diffusive sources. The thermoelastic theories with mass diffusion developed by Sherief et al. [1] and kumar and Kansal [2] have been used to investigate the problem. Laplace and Hankel transforms technique is applied to obtain the solutions of the governing equations. The displacements, stress components, temperature change and chemical potential are obtained in the transformed domain. Numerical inversion technique has been used to obtain the solutions in the physical domain. Effects of couple stress on the resulting quantities are shown graphically. Some particular cases of interest are also deduced.

Keywords : Modified couple stress; Axisymmetric heat sources; Heat generation; Laplace and Hankel transforms.

1 INTRODUCTION

CLASSICAL first gradient approaches in continuum mechanics do not address the size dependency that is observed in smaller scales. Consequently, a number of theories that include higher gradients of deformation have been proposed to capture, at least partially, size-effects at the nano-scale. Additionally, consideration of the second gradient of deformation leads naturally to the introduction of the concept of couple-stresses. Thus, in the current form of these theories, the material continuum may respond to body and surface couples, as well as spin inertia for dynamical problems. The existence of couple-stress in materials was originally postulated by Voigt [3]. However, Cosserat [4] were the first to develop a mathematical model to analyze materials with couple stresses. Lacking an internal material length scale parameter, classical elasticity and plasticity cannot be used to interpret the size effect observed in numerous tests at micron and nanometer scales. However, higher-order (non-local) continuum theories contain material length scale parameters and are capable of explaining microstructure related size (and other effects). Couple stress theories represent one class of such higher-order theories. The classical couple stress elasticity theory was proposed by (e.g., Mindlin and Tiersten [5], Toupin [6], Koiter [7] contains four material constants two classical and two additional for isotropic elastic materials. The couple stress theory can be viewed as a special format of strain gradient theory which uses rotation as a variable to describe curvature, while the strain gradient theory uses strain as variable to describe curvature. Couple-stress theory is an extended continuum theory that includes the effects of a couple per unit area on a material volume, in addition to the classical direct and shear forces per unit area. This immediately admits the possibility of asymmetric stress tensor, since shear stress no longer have to be conjugate in order to ensure rotational equilibrium. The two additional constants are related to the

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underlying microstructure of the material and are inherently difficult to determine (e.g., Lakes [8] Lam et al. [9]. Every physical theory possesses a certain domain of applicability outside which it fails to predict the physical phenomena with reasonable accuracy. Hence, there has been a need to develop higher-order theories involving only one additional material length scale parameter. The small length scale involved in microstructures has questioned the applicability of the classical mechanics model. The small size of the material structure, such as the lattice space between single atoms, is very important in nanotechnology problems. As this scale is ignored in the classical mechanics model, the modified couple-stress theory which was developed by Yang et al.[10], Park and Gao [11]. studied the Bernoulli- Euler beam model based on a modified couple stress theory. Simsek and Reddy [12] investigated the bending and vibration of functionally graded microbeams using a new higher order beam theory and the modified couple stress theory. Recently, Shaat et al.[13] studied the size-dependent bending analysis of Kirchhoff nano-plates based on a modified couple-stress theory including surface effects. Ghorbanpour Arani et al. [14] discussed the problem of vibration of bioliquid-filled microtubules using modified couple stress theory. In this problem, the modified couple stress theory is applied to consider the small scale effects while motion equations are derived using energy method and Hamilton's principle for both Euler-Bernoulli beam (EBB) and Timoshenko beam (TB) models. Darijani and Shahdadi[15] investigated the effect of shear deformation on the static bending and vibration responses of a simply supported microplate by using modified couple stress theory and the governing equations and related boundary conditions are solved simultaneously using Hamilton's principle.

Recently, Wang et al. [16] presented a nonlinear bending and post-buckling of extensible microscalebeams based on modified couple stress theory where the effects of the material length scale parameter and the Poisson ratio on the bending and thermal post-buckling behaviors of microbeams are discussed in detail and the size dependent governing differential equations are solved numerically using shooting method. Diffusion plays an important role in geophysics, metal oxide semiconductor improvement in crude oil extraction from oil deposits. In present, diffusion imaging is essentially used for brain exploration in clinical practice. Nevertheless, new applications are emerging outside neuroradiology (cancerology, musculoskeletal radiology…) etc. Diffusion can be defined as the movement of molecules from a high concentration to a low concentration. Thermodiffusion in an elastic solid is due to the coupling of the fields of temperature, mass diffusion and strain. Heat and mass exchange with the environment during the process of the thermodiffusion in an elastic solid. The concept of thermodiffusion is used to describe the processes of thermomechanical treatment of metals (carboning, nitriding steel, etc.) and these processes are thermally activated, and their diffusing substances being, e.g. nitrogen, carbon etc. They are accompanied by deformations of the solid.

Podstrigach [17] and Nowacki [18-21] developed the theories of thermodiffusion elastic solid in which the coupled thermoelastic model is used and implies infinite speeds of propagation of thermoelastic waves. Sherief et al.[1] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. Sherief and Saleh [22] worked on a problem of a thermoelastic half space with a permeating substance in contact with the bounding plane in the context of the theory of generalized thermoelastic diffusion with one relaxation time. Recently, Kumar and Kansal [2] derived the basic equations in generalized thermoelastic diffusion for Green Lindsay (GL-model) theory and discussed the Lamb waves. El-Maghraby and Abdel-Halim [23] studied a problem of generalized thermoelasticity in Lord and Shulman [24] theory for a half space subjected to a known axisymmetric temperature distributions by using Laplace and Hankel transforms technique. Tripathi et al. [25] investigated the temperature distribution and thermal stresses in a semi-infinite cylinder with heat sources in thermoelastic theory with one relaxation time. Recently, Tripathi et al. [26] discussed the problem of a thick circular plate with axisymmetric heat supply in a generalized thermoelastic diffusion by using integral transform technique.

The objective of this paper is to study the two dimensional axisymmetric problem of thick circular plate in modified couple stress theory with heat sources by applying integral transform technique. The normal stress, tangential stress, couple stress, temperature change and chemical potential are computed and presented graphically for different values of radial distance. Some particular cases are also derived from the present investigation.

2 GOVERNING EQUATIONS

Following (Yang et al.[10] Kumar and Kansal [2]) the constitutive relations and the equations of motion in a modified couple-stress generalized thermoelastic elastic with mass diffusion in the absence of body forces, body couples and mass diffusion sources are given by

Constitutive relations

$$t_{ij} = \lambda e_{kk}\,\delta_{ij} + 2\mu e_{ij} - \frac{1}{2}\,\mathrm{e}_{kij}\mathrm{m}_{lk,l} - \beta_1\left(1+\tau_1\frac{\partial}{\partial t}\right)\mathrm{T}\delta_{ij} - \beta_2\left(1+\tau^1\frac{\partial}{\partial t}\right)\mathrm{C}\delta_{ij}, \tag{1}$$

$$m_{ij} = 2\alpha\chi_{ij}, \tag{2}$$

$$x_{ij} = \frac{1}{2}\left(\omega_{ij} + \omega_{ji}\right), \tag{3}$$

$$\omega_i = \frac{1}{2}\varepsilon_{ipq}u_{q,p}, \tag{4}$$

$$P = -\beta_2 e_{kk} + b\left(1+\tau^1\frac{\partial}{\partial t}\right)C - a\left(1+\tau_1\frac{\partial}{\partial t}\right)T, \tag{5}$$

Equations of motion

$$\left(\lambda+\mu+\frac{\alpha}{4}\Delta\right)\nabla\left(\nabla\vec{u}\right) + \left(\mu-\frac{\alpha}{4}\Delta\right)\nabla^2\vec{u} - \beta_1\left(1+\tau_1\frac{\partial}{\partial t}\right)\nabla T - \beta_2\left(1+\tau^1\frac{\partial}{\partial t}\right)\nabla C = \rho\ddot{\vec{u}}, \tag{6}$$

Equation of heat conduction

$$K^*\Delta T - \rho c_e\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)T - aT_0\left(\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)C + \left(1+\tau_0\frac{\partial}{\partial t}\right)Q = T_0\beta_1\left(\frac{\partial}{\partial t}+\tau_0\eta_0\frac{\partial^2}{\partial t^2}\right)\left(\nabla\vec{u}\right), \tag{7}$$

Equation of mass diffusion

$$D\beta_2\Delta\left(\nabla\vec{u}\right) + Da\left(1+\tau_1\frac{\partial}{\partial t}\right)\Delta T + \left(\frac{\partial}{\partial t}+\tau^0\eta_0\frac{\partial^2}{\partial t^2}\right)C - Db\Delta\left(1+\tau^1\frac{\partial}{\partial t}\right)C = 0, \tag{8}$$

where t_{ij} are the components of stress tensor, λ and μ are material constants, δ_{ij} is Kronecker's delta, e_{ij} are the components of strain tensor, e_{ijk} is alternate tensor, m_{ij} are the components of couple-stress, $\beta_1 = (3\lambda+2\mu)\alpha_t$, $\beta_2 = (3\lambda+2\mu)\alpha_c$, Here α_t, α_c are the coefficients of linear thermal expansion and diffusion expansion respectively, T is the temperature change, C is the mass concentration, α is the couple stress parameter, x_{ij} is symmetric curvature, ω_i is the rotational vector, P is the chemical potential of the material per unit mass, b is the coefficient describing the measure of mass diffusion effects, a is the coefficient describing the measure of thermoelastic diffusion. $\vec{u} = (u_1, u_2, u_3)$ is the components of displacement vector, ρ is the density, Δ is the Laplacian operator, ∇ is del operator. K^* is the coefficient of the thermal conductivity, c_e is the specific heat at constant strain, Q is the heat source, T_0 is the reference temperature assumed to be such that $T/T_0 \prec\prec 1$. D is the thermoelastic diffusion constant, Here τ^0, τ^1 are the diffusion relaxation times with $\tau^1 \geq \tau^0 \geq 0$ and τ_0, τ_1 are thermal relaxation times with $\tau_1 \geq \tau_0 \geq 0$. Here, $\tau_1 = \tau^1 = 0, \eta_0 = 1, \gamma = \tau_0$, for Lord-Shulman (L-S) model and $\eta_0 = 0, \gamma = \tau^0$, for Green Lindsay (G-L) model.

3 FORMULATION OF THE PROBLEM

Consider an axisymmetric homogeneous isotropic, modified couple stress generalized thermodiffusion elastic thick plate of thickness $2d$ defined by the region $0 \le r \le \infty, -d \le z \le d$. Cylindrical polar coordinates (r,ϕ,z) having origin on the surface $z=0$, between the lower and upper surfaces of the plate and the *z*-axis is assumed to be the axis of symmetry. Due to symmetry about *z*-axis all the field quantities depending only on (r,z,t).

The initial temperature in the thick plate is given by a constant temperature T_0 and the heat flux $g_0 F(r,z)$ is prescribed on the upper and lower boundary surfaces. For $t \succ 0$, heat is generated within the plate at the rate $Q(r,z,t)$. Under these conditions, thermoelastic quantities in a semi-infinite thick circular plate are required to be determined. For the two-dimensional problem, we take the displacement vector $\vec{u}=(u_r,0,u_z)$. We define the dimensionless quantities:

$$r'=\frac{\omega^*}{c_1}r, z'=\frac{\omega^*}{c_1}z, u_r'=\frac{\omega^*}{c_1}u_r, u_z'=\frac{\omega^*}{c_1}u_z, t'=\omega^* t, t_{ij}'=\frac{t_{ij}}{\beta_1 T_0}, m_{ij}'=\frac{\omega^* m_{ij}}{c_1\beta_1 T_0}, \gamma'=\omega^*\gamma, \tau_1'=\omega^*\tau_1,$$
$$\tau_0'=\omega^*\tau_0, \tau^{0'}=\omega^*\tau^0, \tau^{1'}=\omega^*\tau^1, T'=\frac{\beta_1 T}{\rho c_1^2}, C'=\frac{\beta_2 C}{\rho c_1^2}, P'=\frac{P}{\beta_2}, Q'=\frac{c_e Q}{K^*\omega^{*2}}, c_1^2=\frac{\lambda+2\mu}{\rho}, \omega^{*2}=\frac{\lambda}{\left(\mu t^2+\rho\alpha\right)}. \tag{9}$$

Upon introducing (9) in Eqs. (6)-(8), after suppressing the primes, we obtain

$$a_1\frac{\partial e}{\partial r}+a_2\left(\nabla^2-\frac{1}{r^2}\right)u_r+a_3\Delta\left(\frac{\partial e}{\partial r}-\left(\nabla^2-\frac{1}{r^2}\right)u_r\right)-\tau_t\frac{\partial T}{\partial r}-\tau_t^1\frac{\partial C}{\partial r}=\frac{\partial^2 u_r}{\partial t^2}, \tag{10}$$

$$a_1\frac{\partial e}{\partial z}+a_2\nabla^2 u_z+a_3\Delta\left(\frac{\partial e}{\partial z}-\nabla^2 u_z\right)-\tau_t\frac{\partial T}{\partial z}-\tau_t^1\frac{\partial C}{\partial z}=\frac{\partial^2 u_z}{\partial t^2}, \tag{11}$$

$$\nabla^2 T-a_4\tau_t^0 T-a_5\tau_\gamma^0 C+a_6\tau_t^{10}Q=a_7\tau_{\eta_0}^0 e, \tag{12}$$

$$a_8\nabla^2 e+a_9\tau_t\nabla^2 T+\tau_t^{20}C-a_{10}\tau_t^1\nabla^2 C=0, \tag{13}$$

where

$$a_1=\frac{(\lambda+\mu)}{\rho c_1^2}, a_2=\frac{\mu}{\rho c_1^2}, a_3=\frac{\alpha\omega^{*2}}{4\rho c_1^4}, a_4=\frac{\rho c_e c_1^2}{K^*\omega^*}, a_5=\frac{aT_0\beta_1 c_1^2}{\beta_2 K^*\omega^*}, a_6=\frac{\beta_1}{\rho c_e}, a_7=\frac{T_0\beta_1^2}{\rho K^*\omega^*}, a_8=\frac{\beta_2^2 D\omega^*}{\rho c_1^4},$$
$$a_9=\frac{a\beta_2 D\omega^*}{\beta_1 c_1^2}, a_{10}=\frac{bD\omega^*}{c_1^2}, \tau_t=\left(1+\tau_1\frac{\partial}{\partial t}\right), \tau_t^1=\left(1+\tau^1\frac{\partial}{\partial t}\right), \tau_t^0=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right), \tau_{\eta_0}^0=\left(\frac{\partial}{\partial t}+\eta_0\tau_0\frac{\partial^2}{\partial t^2}\right),$$
$$\tau_\gamma^0=\left(\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right), \tau_t^{10}=\left(1-\tau_0\frac{\partial}{\partial t}\right), \tau_t^{20}=\left(\frac{\partial}{\partial t}+\eta_0\tau^0\frac{\partial^2}{\partial t^2}\right), \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},$$
$$e=\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial u_z}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z},$$

By Helmholtz theorem, the displacement vector $\vec{u}$ can be expressed as:

$$\vec{u}=\nabla\phi+\nabla\times\vec{\psi} \tag{14}$$

where two potentials ϕ and $\vec{\psi}$ are the Lame's potentials representing irrotational and rotational parts of the displacement vector $\vec{u}$ respectively. It is possible to take only one component of the vector $\vec{\psi}$ to be non-zero, i. e. $\vec{\psi}$ can be written as:

$$\vec{\psi} = \frac{\partial \psi}{\partial r}$$

The displacement components u_r and u_z in terms of potential functions ϕ and $\vec{\psi}$ in dimensionless form are given by

$$u_r = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z}, \tag{15}$$

$$u_z = \frac{\partial \phi}{\partial z} - \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right). \tag{16}$$

with the aid of (15) and (16), Eqs. (10)-(13) yield

$$\left[\delta \nabla^2 - \frac{\partial^2}{\partial t^2} \right] \phi - \tau_t T - \tau_t^1 C = 0, \tag{17}$$

$$\left[a_2 \nabla^2 - a_3 \nabla^4 - \frac{\partial^2}{\partial t^2} \right] \psi = 0, \tag{18}$$

$$a_7 \tau_{\eta_0}^0 \nabla^2 \phi - \left(\nabla^2 - a_4 \tau_t^0 \right) T + a_5 \tau_\gamma^0 C = a_6 \tau_t^{10} Q, \tag{19}$$

$$a_8 \nabla^4 \phi + a_9 \tau_t \nabla^2 T + \left(\tau_t^{20} - a_{10} \tau_t^1 \nabla^2 \right) C = 0, \tag{20}$$

where

$$\delta = \left(a_1 + a_2 \right), e = \nabla^2 \phi$$

We define Laplace and Hankel transforms as:

$$\bar{f}\left(r,z,s\right) = \int_0^\infty f\left(r,z,t\right) e^{-st} dt \qquad \hat{f}\left(\eta,z,s\right) = H\left[\bar{f}\left(r,z,s\right)\right] = \int_0^\infty \bar{f}\left(r,z,s\right) r J_n\left(\eta r\right) dr \tag{21}$$

where S is the Laplace transform parameter, η is the Hankel transform parameter and $J_n()$ is the Bessel function of the first kind of order n. Applying the Laplace and Hankel transforms defined by (21) on Eqs. (17)-(20), after simplification, we obtain

$$\left[D^6 + G_1 D^4 + G_2 D^2 + G_3 \right] \left(\hat{\phi}, \hat{T}, \hat{C} \right) = a_6 \tau_t^{10} Q, \tag{22}$$

$$\left[D^4 - B_1 D^2 + B_2 \right] \hat{\psi} = 0, \tag{23}$$

where

$$G = a_8 + \delta a_{10}\tau_t^{22},$$

$$G_1 = \frac{\begin{Bmatrix} -\left(\eta^2\delta + s^2\right)a_{10}\tau_t^{22} - \delta\left(\tau_t^{77} + a_5 a_9 \tau_t^{11}\tau_t^{44} + \left(a_4\tau_t^{33} + 2\eta^2\right)a_{10}\tau_t^{22}\right) - \tau_t^{11}\left(a_7 a_{10}\tau_t^{22}\tau_t^{55} + a_5 a_8 \tau_t^{44}\right) \\ -\tau_t^{22}\left(a_7 a_8 \tau_t^{11}\tau_t^{55} - a_4 a_8 \tau_t^{33}\right) - 3a_8\eta^2 \end{Bmatrix}}{G},$$

$$G_2 = \frac{\begin{Bmatrix} \left(\eta^2\delta + s^2\right)\left(\tau_t^{77} + a_5 a_9\tau_t^{11}\tau_t^{44} + \left(a_4\tau_t^{33} + 2\eta^2\right)a_{10}\tau_t^{22}\right) + \delta\begin{pmatrix} a_{10}\tau_t^{22}\eta^4 + \\ \left(\tau_t^{77} + a_5 a_9 \tau_t^{11}\tau_t^{44} + a_4 a_{10}\tau_t^{22}\tau_t^{33}\right)\eta^2 + a_4\tau_t^{33}\tau_t^{77} \end{pmatrix} + a_7\tau_t^{11}\tau_t^{55}\tau_t^{77} \\ +2\eta^2\left(a_7 a_{10}\tau_t^{22}\tau_t^{55} + a_5 a_8\tau_t^{44} + a_8\tau_t^{22}\left(a_7\tau_t^{11}\tau_t^{55} - a_4\tau_t^{33}\right)\right) + 3\eta^2 a_8 \end{Bmatrix}}{G},$$

$$G_3 = \frac{\begin{Bmatrix} -\left(\eta^2\delta + s^2\right)\left(a_{10}\tau_t^{22}\eta^4 + \left(\tau_t^{77} + a_5 a_9\tau_t^{11}\tau_t^{44} + a_4 a_{10}\tau_t^{22}\tau_t^{33}\right)\eta^2 + a_4\tau_t^{33}\tau_t^{77}\right) - a_7\tau_t^{11}\tau_t^{55}\tau_t^{77}\eta^2 \\ -\eta^4\left(a_7 a_{10}\tau_t^{22}\tau_t^{55} + a_5 a_8\tau_t^{44} + \tau_t^{22}\left(a_7 a_8\tau_t^{11}\tau_t^{55} - a_4 a_8\tau_t^{33} - a_8\eta^6\right)\right) \end{Bmatrix}}{G},$$

$$B_1 = \frac{a_2 + 2a_3\eta^2}{a_3}, B_2 = \frac{a_3\eta^4 + a_2\eta^2 + s^2}{a_3}, \tau_t^{11} = \left(1 + \tau_1 s\right), \tau_t^{22} = \left(1 + \tau^1 s\right), \tau_t^{33} = \left(s + \tau_0 s^2\right),$$

$$\tau_t^{44} = \left(s + \gamma s^2\right), \tau_t^{55} = \left(s + \eta_0\tau_0 s^2\right), \tau_t^{66} = \left(1 - \tau_0 s\right), \tau_t^{77} = \left(s + \eta_0\tau^0 s^2\right)$$

The general solution of Eq. (22) can be written as:

$$\hat{\phi} = \hat{\phi}_1 + \hat{\phi}_2 + \hat{\phi}_3 + \hat{\phi}_p, \tag{24}$$

where $\hat{\phi}_i\ (i = 1,2,3)$ is a general solution of the homogeneous differential equation given by

$$\left(D^2 - m_i^2\right)\hat{\phi}_i = 0, \qquad i = 1,\ 2,\ 3 \tag{25}$$

The general solution of the Eq. (25) can be written as:

$$\hat{\phi}_i = \sum_{i=1}^{3} A_i \cosh\left(m_i z\right), \tag{26}$$

where m_1, m_2 and m_3 are the roots of the characteristic equation given by

$$[D^6 + G_1 D^4 + G_2 D^2 + G_3] = 0. \tag{27}$$

Also $\hat{\phi}_p$ is the particular solution satisfying the equation

$$\left(D^2 - m_1^2\right)\left(D^2 - m_2^2\right)\left(D^2 - m_3^2\right)\hat{\phi}_p = a_6\tau_t^{66}\hat{Q}, \tag{28}$$

Let the heat generation $Q\left(r,z,t\right)$ be taken as:

$$Q\left(r,z,t\right) = \frac{q_0\delta\left(t\right)\delta\left(r\right)\cosh\left(z\right)}{2\pi r}, \tag{29}$$

This is a cylindrical shell heat source releasing heat instantaneously at $t = 0$ and situated at the centre $r = 0$ varying in the axial direction where q_0 is the strength of the heat generation. Applying Laplace and Hankel transforms defined by (21) on (29), yield

$$\hat{Q} = Q_0 \cosh(z), \tag{30}$$

where $Q_0 = \dfrac{q_0}{2\pi}$.

The solution of the Eq. (28) take the form

$$\hat{\phi}_p = \frac{a_6 \tau_t^{66} Q_0}{\left(1-m_1^2\right)\left(1-m_2^2\right)\left(1-m_3^2\right)} \cosh(z), \tag{31}$$

The complete solution of Eq. (24) can be written as:

$$\hat{\phi} = \sum_{i=1}^{3} A_i \cosh(m_i z) + \frac{a_6 \tau_t^{66} Q_0}{\left(1-m_1^2\right)\left(1-m_2^2\right)\left(1-m_3^2\right)} \cosh(z), \tag{32}$$

Solving Eq. (23), we get

$$\hat{\psi} = \hat{\psi}_4 + \hat{\psi}_5 \tag{33}$$

where $\hat{\psi}_i \ (i = 4,5)$ is a solution of the homogeneous differential equation given by

$$\left(D^2 - m_i^2\right)\hat{\psi}_i = 0, \qquad i = 4,5 \tag{34}$$

The solution of the Eq. (23), yield

$$\hat{\psi} = \sum_{i=4}^{5} A_i \sinh(m_i z), \tag{35}$$

where m_4 and m_5 are the roots of the characteristic Eq. (23). Similarly, the solution of the Eq. (22) is

$$\left(\hat{\phi}, \hat{T}, \hat{C}\right)(\eta, z, s) = \sum_{i=1}^{3} \left(1, R_i, S_i\right) \left(A_i \cosh(m_i z) + \frac{a_6 \tau_t^{66} Q_0}{\left(1-m_1^2\right)\left(1-m_2^2\right)\left(1-m_3^2\right)} \cosh(z) \right), \tag{36}$$

where

$$R_i = \sum_{i=1}^{3} \frac{a_7 \tau_t^{55}\left(m_i^2 - \eta^2\right)\left(\tau_t^{77} - a_{10}\left(m_i^2 - \eta^2\right)\tau_t^{22}\right) - a_5 a_8 \tau_t^{44}\left(m_i^2 - \eta^2\right)^2}{\left[\left(-\left(m_i^2 - \eta^2\right) + a_4 \tau_t^{33}\right)\left(\tau_t^{77} - \left(m_i^2 - \eta^2\right)\left(a_{10}\tau_t^{22} + a_5 a_9 \tau_t^{11} \tau_t^{44}\right)\right)\right]},$$

$$S_i = \sum_{i=1}^{3} \frac{\left(m_i^2 - \eta^2\right)^2 \left[a_7 a_9 \tau_t^{11} \tau_t^{55} + a_8\left(\left(m_i^2 - \eta^2\right) - a_4 \tau_t^{33}\right)\right]}{\left[\left(-\left(m_i^2 - \eta^2\right) + a_4 \tau_t^{33}\right)\left(\tau_t^{77} - \left(m_i^2 - \eta^2\right)\left(a_{10}\tau_t^{22} + a_5 a_9 \tau_t^{11} \tau_t^{44}\right)\right)\right]}, \qquad i = 1,2,3$$

4 BOUNDARY CONDITIONS

$$\frac{\partial T}{\partial z} = \pm g_0 F(r,z) \qquad at \qquad z = \pm d, \tag{37}$$

$$t_{zz} = t_{zr} = m_{z\phi} = 0 \qquad at \qquad z = \pm d, \tag{38}$$

$$P = \delta(t) f(r) \qquad at \qquad z = \pm d, \tag{39}$$

where

$$F(r,z) = z^2 e^{-\omega r}, \ \ \omega \succ 0. \tag{40}$$

$$f(r) = H(a-r) \tag{41}$$

and $\delta()$ is the Dirac delta function, H is the Heavy side unit step function. Applying Laplace and Hankel transforms defined by (21) on (40) and (41), we obtain

$$\hat{F}(\eta,z) = \frac{z^2\omega}{\left(\omega^2+\alpha^2\right)^{3/2}}, \tag{42}$$

$$\hat{f}(\eta) = \frac{aJ_1(\eta a)}{\eta}, \tag{43}$$

The non-dimensional values of $t_{zz}, t_{zr}, m_{z\phi}$ and P are given by (1)-(5) and (9), as:

$$t_{zz} = a_{11}e + 2a_{12}\left(\frac{\partial u_z}{\partial z}\right) - a_{13}\left\{\left(1+\tau_1\frac{\partial}{\partial t}\right)T + \left(1+\tau^1\frac{\partial}{\partial t}\right)C\right\}, \tag{44}$$

$$t_{zr} = a_{12}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) - a_{14}\left[\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\right], \tag{45}$$

$$m_{z\phi} = 2a_{14}\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{\partial^2 u_z}{\partial r \partial z}\right), \tag{46}$$

$$P = -e + a_{15}\tau_t^1 C - a_{16}\tau_t T, \tag{47}$$

where

$$a_{11} = \frac{\lambda}{\beta_1 T_0}, a_{12} = \frac{\mu}{\beta_1 T_0}, a_{13} = \frac{\rho c_1^2}{\beta_1 T_0}, a_{14} = \frac{\alpha\omega^{*2}}{4c_1^2\beta_1 T_0}, a_{15} = \frac{b\rho c_1^2}{\beta_2^2}, a_{16} = \frac{\alpha\rho c_1^2}{\beta_1\beta_2}.$$

Substituting the values of $\hat{\phi}, \hat{T}, \hat{C}$ and $\hat{\psi}$ from (35) and (36) in the boundary conditions (37)-(39) and with the aid of (15), (16), (21) and (40)-(47), we obtain the expressions for displacement components, stresses, temperature change and chemical potential as:

$$\hat{u}_r = -\eta\left[\sum_{i=1}^{3} A_i \cosh(m_i d) + \sum_{i=4}^{5} A_i \sinh(m_i d) + H\cosh(d)\right], \tag{48}$$

$$\hat{u}_z = \left[\sum_{i=1}^{3} m_i A_i \sinh(m_i d) + \eta^2 \sum_{i=4}^{5} A_i \cosh(m_i d) + H\sinh(d)\right], \tag{49}$$

$$\hat{T} = \left[\sum_{i=1}^{3} R_i A_i \cosh(m_i d) + \sum_{i=1}^{3} R_i H\cosh(d)\right], \tag{50}$$

$$\hat{t}_{zz} = \left[\sum_{i=1}^{3} M_i A_i \cosh(m_i d) + \sum_{i=4}^{5} M_i A_i \cosh(m_i d) + P_2\right], \tag{51}$$

$$\hat{t}_{zr} = -\eta\left[\sum_{i=1}^{3} N_i A_i \sinh(m_i d) + \sum_{i=4}^{5} N_i A_i \sinh(m_i d) + P_3\right], \tag{52}$$

$$\hat{P} = \left[\sum_{i=1}^{3} K_i A_i \cosh(m_i d) + \sum_{i=4}^{5} K_i A_i \cosh(m_i d) + P_4\right], \tag{53}$$

$$\hat{m}_{z\phi} = -2\eta a_{14}\left[\sum_{i=4}^{5} G_i A_i \cosh(m_i d)\right], \tag{54}$$

$$\hat{C} = \left[\sum_{i=1}^{3} S_i A_i \cosh(m_i d) + \sum_{i=1}^{3} S_i H\cosh(d)\right], \tag{55}$$

where

$$A_1 = \frac{\Delta_1}{\Delta}, A_2 = \frac{\Delta_2}{\Delta}, A_3 = \frac{\Delta_3}{\Delta}, A_4 = \frac{\Delta_4}{\Delta}, A_5 = \frac{\Delta_5}{\Delta},$$

$$\begin{aligned}
\Delta = R_1 m_1 g_1 &\begin{bmatrix} M_2 h_2\left(N_3 K_5 g_3 h_5 - N_5 K_3 g_5 h_3 + N_3 K_4 g_3 h_4 - N_4 K_3 g_4 h_3\right) \\ -M_3 h_3\left(N_2 K_5 g_2 h_5 - N_5 K_2 g_5 h_2 + N_2 K_4 g_2 h_4 - N_4 K_2 g_4 h_2\right) \\ +M_4 h_4\left(N_2 K_3 g_2 h_3 - N_3 K_2 g_3 h_2\right) + M_5 h_5\left(N_2 K_3 g_2 h_3 - N_3 K_2 g_3 h_2\right) \\ \end{bmatrix} \\
+R_2 m_2 g_2 &\begin{bmatrix} M_1 h_1\left(N_3 K_5 g_3 h_5 - N_5 K_3 g_5 h_3 - N_3 K_4 g_3 h_4 + N_4 K_3 g_4 h_3\right) \\ -M_3 h_3\left(N_1 K_5 g_1 h_5 - N_5 K_1 g_5 h_1 - N_1 K_4 g_1 h_4 + N_4 K_1 g_4 h_1\right) \\ +M_4 h_4\left(N_3 K_1 g_3 h_1 - N_1 K_3 g_1 h_3\right) + M_5 h_5\left(N_1 K_3 g_1 h_3 - N_3 K_1 g_3 h_1\right) \\ \end{bmatrix} \\
+R_3 m_3 g_3 &\begin{bmatrix} M_1 h_1\left(N_2 K_5 g_2 h_5 - N_5 K_2 g_5 h_2 - N_4 K_2 g_4 h_2 + N_2 K_4 g_2 h_4\right) \\ -M_2 h_2\left(N_1 K_5 g_1 h_5 - N_5 K_1 g_5 h_1 + N_1 K_4 g_1 h_4 - N_4 K_1 g_4 h_1\right) \\ +M_4 h_4\left(N_1 K_2 g_1 h_2 - N_2 K_1 g_2 h_1\right) + M_5 h_5\left(N_1 K_2 g_1 h_2 - N_2 K_1 g_2 h_1\right) \end{bmatrix},
\end{aligned}$$

$$g_1 = \sinh(m_1 d), g_2 = \sinh(m_2 d), g_3 = \sinh(m_3 d), g_4 = \sinh(m_4 d), g_5 = \sinh(m_5 d),$$
$$h_1 = \cosh(m_1 d), h_2 = \cosh(m_2 d), h_3 = \cosh(m_3 d), h_4 = \cosh(m_4 d), h_5 = \cosh(m_5 d),$$

$\Delta_i \left(i = 1,..........,5\right)$ are obtained by replacing 1st, 2nd, 3rd, 4th and 5th column by

$$\left[\left(g_0 \hat{F}(\eta z) - P_1\right), -P_2, -P_3, 0, \left(\hat{F}(\eta) - P_4\right)\right]^T \; in \; \Delta_i$$

And

$$H = \frac{a_6 \tau_t^{66} Q_0}{\left(1-m_1^2\right)\left(1-m_2^2\right)\left(1-m_3^2\right)}, P_1 = \sum_{i=1}^{3} R_i H \sinh(d), P_2 = H\left[a_{11}\left(1+\eta^2\right) + 2a_{12} - a_{13}\left(\tau_t^{11} + \tau_t^{22}\right)\right]\cosh(d),$$

$$P_3 = -2Ha_{12}\eta \sinh(d), P_4 = \left[-\left(1+\eta^2 + \sum_{i=1}^{3}\left(a_{15}\tau_t^{22} S_i - a_{16}\tau_t^{66} R_i\right)\right)\right] H \cosh(d),$$

$$\sum_{i=1}^{3} M_i = \sum_{i=1}^{3}\left[a_{11}\left(m_i^2 - \eta^2\right) + 2a_{12} m_i^2 - a_{13}\left(\tau_t^{11} R_i + \tau_t^{22} S_i\right)\right], \sum_{i=4}^{5} M_i = \sum_{i=4}^{5}\left[2\left(a_{11} + a_{12}\right) m_i \eta^2\right],$$

$$\sum_{i=1}^{3} N_i = \sum_{i=1}^{3} 2a_{12} m_i, \sum_{i=4}^{5} N_i = \sum_{i=4}^{5}\left[a_{12}\left(m_i^2 + \eta^2\right) - a_{14}\left(m_i^2 - \eta^2\right)^2\right], \sum_{i=4}^{5} G_i = \sum_{i=4}^{5} m_i \left(m_i^2 - \eta^2\right),$$

$$\sum_{i=1}^{3} K_i = \sum_{i=1}^{3}\left[-\left(m_i^2 + \eta^2\right) + a_{15}\tau_t^{22} S_i - a_{16}\tau_t^{66} R_i\right], \sum_{i=4}^{5} K_i = \sum_{i=4}^{5} -2\eta^2 m_i,$$

5 PARTICULAR CASES

If $\alpha = 0$, in Eqs. (48)-(55), we obtain the components of displacement and stresses for a generalized thermoelastic with mass diffusion medium. The results obtained are similar as given by [26] with the changed value of

$$\hat{F}(\eta, z) = \frac{z^2 \omega}{\left(\omega^2 + \alpha^2\right)^{3/2}}$$

In the absence of diffusion $\left(a = D = \tau_t^1 = 0\right)$, in Eqs. (48)-(55), we obtain the components of displacement and stresses in a modified couple stress thermoelastic medium.

If $\tau_1 = \tau^1 = 0, \eta_0 = 1, \gamma = \tau_0$, in Eqs. (48)-(55), we obtain the corresponding results for modified couple stress thermoelastic with mass diffusion for Lord Shulman (L-S) model.

If $\eta_0 = 0, \gamma = \tau^0$, in Eqs. (48)-(55), we obtain the corresponding results for modified couple stress thermoelastic with mass diffusion for Green Lindsay (G-L) model.

6 NUMERICAL INVERSION OF THE TRANSFORMS

To obtain the solution of the problem in physical domain, we must invert the transforms in (48)-(55) for all the theories. Here the displacement components, normal and tangential stresses, temperature change, chemical potential and mass concentration are functions of *Z*, the parameters of Laplace and Hankel transforms *S* and η respectively and hence are of the form $\hat{f}(\eta,z,s)$. We first invert the Hankel transform, which gives the Laplace transform expression $\bar{f}(r,z,s)$ of the function $f(r,z,t)$ as:

$$\bar{f}(r,z,s)=\int_0^\infty \eta \hat{f}(\eta,z,s)J_n(\eta r)d\eta \tag{56}$$

Now for fixed values of η, r and Z, the function $\bar{f}(r,z,s)$ in (56) can be considered as the Laplace transform $\bar{g}(s)$ of the same function $g(t)$. Following [27], the Laplace transformed function $\bar{g}(s)$ can be inverted as given below:

$$g(t)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st}\bar{g}(s)ds, \tag{57}$$

where C is an arbitrary real number greater than all the real parts of the singularities of $\bar{g}(s)$. Taking $s=C+iy$, we get

$$g(t)=\frac{e^{Ct}}{2\pi}\int_{-\infty}^{\infty} e^{ity}\bar{g}(C+iy)dy, \tag{58}$$

Now, taking $e^{-Ct}g(t)$ as $h(t)$ and expanding it as Fourier series in $[0,2L]$, we obtain approximately the formula [27]

$$g(t)=g_\infty(t)+E_D, \tag{59}$$

$$g_\infty(t)=(C_0/2)+\sum_{k=1}^{\infty}C_k, \quad \text{for} \quad 0\le t\le 2L. \tag{60}$$

and

$$C_k=\left(e^{Ct}/L\right)\mathrm{Re}\left[e^{ik\pi t/L}\,\bar{g}\left(C+(ik\pi/L)\right)\right]. \tag{61}$$

E_D is the discretization error that can be made arbitrarily small by choosing a large enough *C*. The values of *C* and *L* are chosen according to the criteria outlined by [27]. Since the infinite series in (60) can be summed up only to a finite number of *N* terms, the approximate value of $g(t)$ becomes

$$g_N(t)=(C_0/2)+\sum_{k=1}^{N}C_k, \quad \text{for} \quad 0\le t\le 2L. \tag{62}$$

We now introduce a truncation error E_T that must be added to the discretization error to produce the total approximate error in evaluating $g(t)$ using the above formula. Two methods are used to reduce the total error. The discretization errors is reduced by using the 'Korrecktur' method [27] and then the 'ε-algorithm' is used to reduce the truncation error and hence to accelerate the convergence. The Korrecktur method formula, to evaluate the function $g(t)$ is

$$g(t) = g_\infty(t) - e^{-2CL} g_\infty(2L+t) + E_D^{'},$$

where $\left|E_D^{'}\right| \prec\prec \left|E_D\right|$ (Honig & Hirdas [27]). Thus the approximate value of $g(t)$ becomes

$$g_{N_k}(t) = g_N(t) - e^{-2CL} g_{N^{'}}(2L+t), \tag{63}$$

where $N^{'}$ is an integer such that $N^{'} < N$. We shall now describe the ε-algorithm, which is used to accelerate the convergence of the series in (62). Let N be an odd natural number and $s_m = \sum_{k=1}^{m} C_k$ be the sequence of partial sums of (62). We define the ε-sequence by

$$\varepsilon_{0,m} = 0, \quad \varepsilon_{1,m} = s_m, \quad \varepsilon_{n+1,m}\varepsilon_{n-1,m+1} + \frac{1}{\varepsilon_{n,m+1} - \varepsilon_{n,m}}; \quad n,m = 1,2,3.......$$

It can be shown that [27], the sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, ..., \varepsilon_{N,1}$ converges to $g(t) + E_D - (C_0/2)$ faster than the sequence of partial sums $s_m, m = 1,2,3......$.

The actual procedure to invert the Laplace transform consists of (63) together with the ε-algorithm. The last step is to calculate the integral in Eq. (56). The method for calculating this integral is described by [28]. It involves the use of Romberg's integration with adaptive step size. This also uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

7 NUMERICAL RESULTS AND DISCUSSION

For numerical computations, following [22], we take the copper material (thermoelastic diffusion solid) as:

$$\lambda = 7.76\times10^{10}\ \text{Kg m}^{-1}\ \text{s}^{-2}, \mu = 3.86\times10^{10}\ \text{Kg m}^{-1}\ \text{s}^{-2}, T_0 = 0.293\times10^{3}\ \text{K}, a = 1.02\times10^{4}\ \text{m}^2\text{s}^{-2}\text{K}^{-1},$$
$$c_e = 0.3831\times10^{3}\ \text{J Kg}^{-1}\ \text{K}^{-1}, \alpha_t = 1.78\times10^{-5}\ \text{K}^{-1}, \alpha_c = 1.98\times10^{-4}\text{m}^3\ \text{Kg}^{-1}, b = 9\times10^{5}\ \text{Kg}^{-1}\text{m}^5\text{s}^{-2},$$
$$D = 0.85\times10^{-8}\ \text{Kg s m}^{-3}, \rho = 8.954\times10^{3}\ \text{Kg m}^{-3}, K^* = 0.386\times10^{3}\ \text{Wm}^{-1}\text{K}^{-1}, \alpha = .05\ \text{Kg m s}^{-2},$$
$$t = 1s, q_0 = 1, \omega = 10s^{-1}, \tau_0 = 0.01s, \tau^0 = 0.03s, \tau_1 = 0.02s, \tau^1 = 0.04s.$$

The software Matlab 7.10.4 has been used to determine the normal stress, tangential stress, couple stress, temperature change and mass concentration for different values of couple stress for both L-S and G-L theories are computed numerically and shown graphically in Figs. 1-5 respectively.

In Figs. 1-5, solid line (-) corresponds to L-S $\alpha = 0$, solid line with centre symbol(-*-) corresponds to L-S $\alpha = 0.05$. Similarly, small dash line (----) corresponds to G-L $\alpha = 0$ and small dash line with centre symbol (---*---) corresponds to G-L $\alpha = 0.05$ respectively.

Fig.1 shows that the variations of normal stress with radial distance. The values of normal stress decreases rapidly as distance increases in the whole range for both values of α and both L-S and G-L theories. On the other hand, the values of t_{zz} for L-S theory is higher in comparison to G-L theory for $\alpha = 0.05$ and opposite behavior is noticed for $\alpha = 0$.

Fig. 2 depicts that the variation of tangential stress t_{zr} with radial distance *r*. The values of tangential stress decrease monotonically with high magnitude in the range $0 \le r \le 1.0$ for both theories of thermoelasticity. Also, the values of tangential stress for L-S $(\alpha = 0, 0.05)$ is higher in comparison to G-L $(\alpha = 0, 0.05)$.

Fig. 3 represents that the variation of couple stress $m_{z\phi}$ with *r*. Similar trend is noticed for both the values of α and both theories of thermoeasticity. On the other hand, the values of $m_{z\phi}$ for $\alpha = 0.5$ is smaller in comparison to $\alpha = 0, 0.05$ with small magnitude for both the theories.

Fig. 4 depicts the variation of temperature change *T* with radial distance for $\alpha = 0, 0.05$. It is noticed that the behavior of $\alpha = 0, 0.05$ is similar for both the theories of thermoelastictiy. The values of temperature change for $\alpha = 0$ is more in comparison to $\alpha = 0.05$ for L-S theory and opposite behavior is noticed for G-L theory.

Fig. 5 shows that the variations of chemical potential *P* with *r*. It is observed that the values of chemical potential decrease monotonically in the whole range for both values of α. The effect of Couple stress increases the values of chemical potential *P* for both L-S and G-L theories.

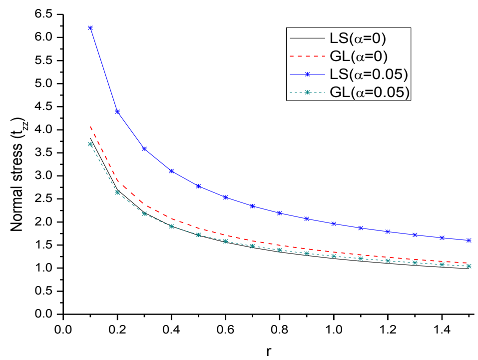

Fig.1
Variation of normal stress with radial distance.

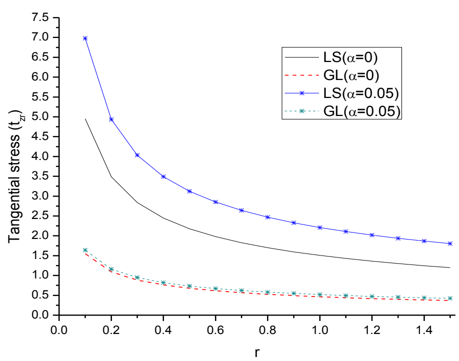

Fig.2
Variation of tangential stress with radial distance.

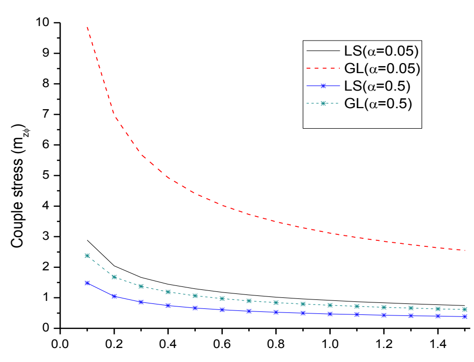

Fig.3
Variation of couple stress with radial distance.

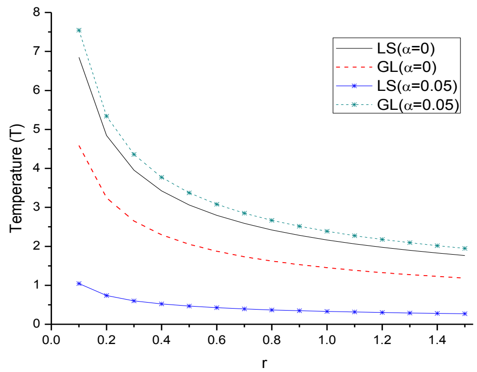

Fig.4
Variation of temperature with radial distance.

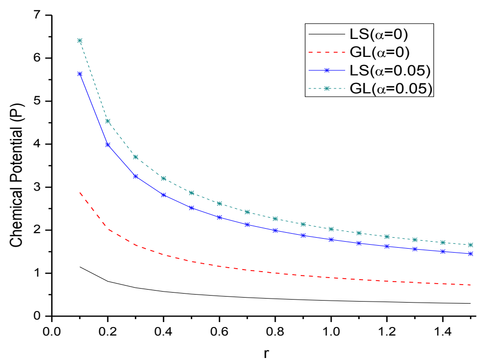

Fig.5
Variation of chemical potential with radial distance.

8 CONCLUSIONS

The problem of thick circular plate with heat generation in modified couple stress thermoelastic diffusion medium is a significant problem of continuum mechanics. The results obtained from above study are summarized as.

The resulting quantities depicted graphically are observed to be very sensitive towards the couple stress parameters. It is evident that the physical quantities are also effected by the different non-classical theories of thermodiffusion elasticity. It is observed that the values of $m_{z\phi}$ and P for G-L theory are more in comparison to L-S theory due to the effect of couple stress and reverse behavior is observed for t_{zr}. Similarly, it is noticed that couple stress increases the values of t_{zz} for L-S theory and decreases the values of temperature for G-L theory.

The results obtained in the study should be beneficial for people working on modified couple stress with heat sources.

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