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Torsional Surface Wave Propagation in Anisotropic Layer Sandwiched Between Heterogeneous Half-Space

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ABSTRACT

The present paper studies the possibility of propagation of torsional surface waves in an inhomogeneous anisotropic layer lying between two heterogeneous half-spaces (upper and lower half-space). Both the half-spaces are assumed to be under compressive initial stress. The study reveals that under the assumed conditions, a torsional surface wave propagates in the medium. The dispersion relation of torsional surface wave has been obtained in the presence of heterogeneity, initial stress and anisotropic, and it is observed that the inhomogeneity factor due to quadratic and hyperbolic variations in rigidity, density and initial stress of the medium decreases the phase velocity as it increases. The result also shows that the initial stresses have a pronounced influence on the propagation of torsional surface waves. In the absence of anisotropy, Initial stress, inhomogeneity and rigidity of the upper half-space, then the dispersion relation coincide with the classical dispersion relation of Love wave.

Keywords : Torsional wave; Heterogeneity; Initial stress; Phase velocity; Dispersion relation.

1 INTRODUCTION

THE propagation of surface waves in an anisotropic layer has importance to the seismologists due to its possible applications in geophysical prospecting and in understanding the cause and estimation of damage due to earthquakes. The particles of the medium twist clockwise and anticlockwise about the direction of motion of the waves during the propagation of torsional surface waves. Due to the importance in various fields such as civil engineering and its several sub-disciplines including architectural engineering, geotechnical engineering, control engineering, structural engineering, earthquake engineering, the study of torsional surface waves in anisotropic layered between two inhomogeneous half spaces has been of central interest to the theoretical seismologists. However, propagation of a torsional waves in such a heterogeneity model has not been studied yet. Earthquakes are often attributed to different types of seismic waves generated at the earthquake focus. Seismic anisotropy is the variation of seismic wave speed with direction. Seismic anisotropy is an indicator of long range order in a material, where features smaller than the seismic wavelength (e.g., crystals, cracks, pores, layers or inclusions) have a dominant alignment. This alignment leads to a directional variation of elasticity wave speed.

In this study, a heterogeneity model is proposed to propagate the torsional surface waves in anisotropic layer lying between two half-spaces (as shown in Fig. (1)). Numerous papers on the propagation of surface waves in inhomogeneous half-space have been published in various journals, due to their devastating damage capabilities

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during earthquake and possible applications in geophysical prospecting. Literature for propagation of torsional surface waves is relatively very less, if compared to other surface waves. Abo-Dahab [1] has studied reflection of *P* and *SV* waves from stress-free surface elastic half-space under influence of magnetic field and hydro-static initial stress without energy dissipation. Abd-Alla and Ahmed [2] subsequently discussed omitted the propagation of Love waves in a nonhomogeneous orthotropic elastic layer under initial stress overlying semi-infinite medium. Propagation of Rayleigh waves in generalized magneto-thermoelastic orthotropic material under initial stress and gravity field was studied by Abd-Alla et al. [3]. Ahmed and Abo-Dahab [4] has shown the propagation of Love waves in an orthotropic granular layer under initial stress overlying a semi-infinite granular medium, also Chattopadhyay et al. [5] described the torsional wave propagation in harmonically inhomogeneous media. Torsional surface waves in heterogeneous anisotropic half-space under initial stress have been studied by Chattopadhyay et al. [6]. Effect of rigid boundary on the propagation of torsional waves in a homogeneous layer over a heterogeneous half-space was formulated by Gupta et al. [7] and effect of rigid boundary on propagation of torsional surface waves in porous elastic layer again improved by Gupta et al. [8].

The effect of initial stress in the medium is due to many reasons, for example resulting from the difference of temperature, process of quenching, shot peening and cold working, slow process of creep, differential external forces, gravity variations etc. These stresses have a pronounced influence on the wave propagation as shown by Biot [9]. Kepceler [10] described the torsional wave dispersion relation in a pre-stressed bi-material compounded cylinder with an imperfect interface and torsional wave propagation in a pre-stressed circular cylinder embedded in a pre-stressed elastic medium was contribute by Ozturk and Akbbarov [11]. Several graphs are plotted to show the nature of torsional surface waves in heterogeneity model, for that purpose the numerical data of the components have been taken from various books (Love [12-13], Ewing et al. [14], Biot [15]).

Many scientists have studied omitted the propagation of various surface waves in heterogeneous medium like: Rayleigh wave, SH-waves, Love waves etc. Khaled et al. [16] calculated homotopy perturbation method and variational iteration method for harmonic wave propagation in nonlinear magneto-thermoelasticity with rotation. Biot [17] discussed omitted the theory of propagation of elastic waves in a fluid saturated porous solid I low frequency range. Recently Kumari et al. [18] studied omitted the propagation of torsional waves in an inhomogeneous layer sandwiched between inhomogeneous semi infinite strata. Selim [19] calculated mathematically the propagation of torsional surface waves in heterogeneous half-space with irregular free surface. Torsional wave propagation in a pre-stressed hyperelastic annular circular cylinder was formulated by Shearer [20]. Propagation of magnetoelastic shear waves in an irregular self-reinforced layer was studied by Chattopadhyay and Singh [21]. Existence of torsional surface waves in an earth's crustal layer lying over a sandy mantle investigated by Vishwakarma and Gupta [22]. Georgiadis et al. [23] have summarized the torsional surface waves in a gradient-elastic half-space. Dey and Dutta [24] concluded the torsional wave propagation in an initially stressed cylinder. Keeping these in view (propagation of various surface waves and torsional surface waves), the present paper contains a heterogeneity model with three types of heterogeneity in the layer and half-spaces. Quadratic and hyperbolic variations in rigidity, density and initial stresses has been taken in upper and lower half-spaces respectively with the inhomoheneity parameters α and β having dimension equal to length. Whereas, middle layer is considered as inhomogeneous anisotropic layer with variation in rigidity, density as $N = N_{20}e^{\beta z}, L = L_{20}e^{\beta z}, \rho = \rho_{20}e^{\beta z}$ where β is inhomogeneity parameter having dimension equal to length. x-axis is taken along the direction of wave propagation and *z*-axis is vertically downward to the direction of wave propagation having origin at the interface (as shown in Fig. (1)). The inhomogeneity and initial stresses of half spaces also have visible effects on the phase velocity of wave propagation. It is observed that phase velocity increases with increase in initial stress of half spaces. The presents model gives the dispersion relation of torsional surface waves in the presences of anisotropic, heterogeneity and initial stress. The classical dispersion relation of Love waves omitted obtained in the absence of anisotropy, initial stresses, inhomogeneity and rigidity L_{10} of the upper half-space, in other words, the torsional wave mode changes into the Love wave mode and this conversion shows that such kind heterogeneity model exists in the Earth and allows the torsional surface waves to propagate.

2 FORMULATION OF THE PROBLEM

A heterogeneity model has been assumed to the propagation of torsional surface waves. The model is well-equipped with an inhomogeneous anisotropic layer of finite thickness H lying between two half-spaces. In this model the upper and lower half-space considered under initial stress with quadratic and hyperbolic variation in rigidities and density respectively. We have described a cylindrical coordinate system with *z*-axis positive vertically downwards. The origin of the coordinate system is located at the surface of the lower half-space at the circular region and *x*-axis is taken along the direction of wave propagation (as shown in Fig. 1).

Variation in rigidity, density and initial stress in upper half-space have been expressed as:

$$N = N_{10}(1+\alpha z), L = L_{10}(1+\alpha z), \rho=\rho_{10}(1+\alpha z), P = P_{10}(1+\alpha z)$$

Similarly, for lower homogeneous half-space

$$N = N_{30}\cosh^2(\gamma z), L = L_{30}\cosh^2(\gamma z), \rho=\rho_0\cosh^2(\gamma z), P = P_0\cosh^2(\gamma z)$$

and anisotropic layer having variation as:

$$N = N_{20}e^{\beta z}, L = L_{20}e^{\beta z}, \rho = \rho_{20}e^{\beta z}$$

where $N_{10}, L_{10}, N_{20}, L_{20}, N_{30}, L_{30}, \rho_{10}, \rho_{20}, \rho_{30}$ and P_{10}, P_{30} represents rigidities, densities and initial stresses in respective mediums.

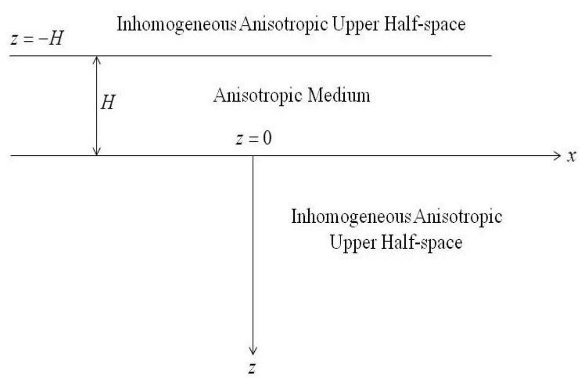

Fig.1
Geometry of the problem.

The following boundary conditions must be satisfied. The continuity of the displacement at interface $z = -H$ is

$$v_0(z) = v_1(z), L_{10}\frac{\partial v_0}{\partial z} = L_{20}\frac{\partial v_1}{\partial z} \tag{1}$$

and continuity at the interface $z = 0$ is

$$v_1(z) = v_2(z), L_{20}\frac{\partial v_1}{\partial z} = L_{30}\frac{\partial v_2}{\partial z} \tag{2}$$

where $v_0(z)$ and $v_2(z)$ are displacement components of upper and lower half-space and $v_1(z)$ is the displacement component of middle layer.

3 SOLUTION OF THE PROBLEM

3.1 Dynamics and solution for the inhomogeneous anisotropic layer

We have considered a cylindrical co-ordinate system with *z*-axis positive downwards. The origin of the coordinate system is located at the surface of the lower half-space at the circular region (as shown in Fig.1).

The equation of motion in an anisotropic layer may be written as Biot [15]:

$$\frac{\partial \sigma_{r\theta}^{(1)}}{\partial r} + \frac{\partial \sigma_{z\theta}^{(1)}}{\partial z} + \frac{2}{r}\sigma_{r\theta}^{(1)} = \rho \frac{\partial^2 v}{\partial t^2} \tag{3}$$

where r and θ be radial and circumferential coordinates respectively, ρ is the density and $v = (r, z, t)$ is the displacement component along θ direction. Stress-strain relation for an inhomogeneous anisotropic elastic layer given as:

$$\sigma_{r\theta}^{(1)} = 2Ne_{r\theta}, \sigma_{z\theta}^{(1)} = 2Ne_{z\theta} \tag{4}$$

where strain components are express as $e_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right), e_{z\theta} = \frac{1}{2}\frac{\partial v}{\partial z}$. N and L are represents rigidities of the medium along r and z directions, respectively. Eq. (3) takes the following form by using relation (4)

$$N\left(\frac{\partial^2 v}{\partial r^2} - \frac{v}{r^2} + \frac{1}{r}\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial z}\left(L\frac{\partial v}{\partial z}\right) = \rho \frac{\partial^2 v}{\partial t^2} \tag{5}$$

when wave propagates along radial direction with amplitude of displacement as a function of depth then the solution of Eq. (5) may be expressed as:

$$v = V(z) J_1(kr) e^{i\omega t} \tag{6}$$

ω represent circular frequency of the wave and J_1 is the Bessel's function of first kind and of order one. $V(z)$ is the solution of following equation

$$\frac{d^2 V}{dz^2} + \frac{1}{L}\frac{dL}{dz}\frac{dV}{dz} - \frac{k^2 N}{L}\left(1 - \frac{c^2 \rho}{N}\right)V = 0 \tag{7}$$

where c=ω/k is the velocity of the propagation of torsional surface wave. Now, substituting, $V = V_1/\sqrt{L}$ in Eq. (7), we get

$$\frac{d^2 V_1}{dz^2} - \frac{1}{2L}\left\{\frac{d^2 L}{dz^2} - \frac{1}{2L}\left(\frac{dL}{dz}\right)^2\right\}V_1 = \frac{k^2 N}{L}\left(1 - \frac{c^2 \rho}{N}\right)V_1 \tag{8}$$

Consider the exponential variation in elastic moduli, rigidities and density in the inhomogeneous anisotropic layer i.e.

$$N = N_{20} e^{\beta z}, L = L_{20} e^{\beta z}, \rho = \rho_{20} e^{\beta z} \tag{9}$$

where β is inhomogeneity parameter and N_{20}, L_{20} and ρ_{20} are rgidities and density at $z \to 0$. Using Eq. (9), Eq. (8) becomes:

$$\frac{d^2 V_1}{dz^2} + \lambda_1^2 V_1 = 0 \tag{10}$$

where $\lambda_1^2 = k^2\left\{N_{20}/L_{20}\left(c^2/c_1^2 - 1\right) - \beta^2/4k^2\right\}$ and $c_1 = \sqrt{N_{20}/\rho_{20}}$ is the shear wave velocity in the layer. The solution of Eq. (8) is given by

$$V_1(z) = A_1 e^{-i\lambda_1 z} + B_1 e^{i\lambda_1 z} \tag{11}$$

where A_1 and B_1 are arbitrary constants and hence the displacement in the middle inhomogeneous anisotropic layer is given by

$$v = v_1 = 1/\sqrt{L_{20}}\left\{A_1 e^{-i\lambda_1 z} + B_1 e^{i\lambda_1 z}\right\} J_1(kr) e^{i\omega t} \tag{12}$$

4 DYNAMICS OF PRE-STRESSED INHOMOGENEOUS ANISOTROPIC HALF-SPACE

If r and θ are the radial and circumferential coordinates respectively, the equation of motion for the initially stressed anisotropic half space is given by Biot [15]:

$$\frac{\partial \sigma_{r\theta}{}^{(2)}}{\partial r} + \frac{\partial \sigma_{z\theta}{}^{(2)}}{\partial z} + \frac{2}{r}\sigma_{r\theta}{}^{(2)} - \frac{\partial}{\partial z}\left(\frac{P}{2}\frac{\partial v}{\partial z}\right) = \rho \frac{\partial^2 v}{\partial t^2} \tag{13}$$

where $\sigma_{r\theta}{}^{(2)}$ and $\sigma_{z\theta}{}^{(2)}$ represents incremental stress components of anisotropic half-spaces, $v = v(r, z, t)$ the displacement along θ direction, *P* is the initial compressive stress along the radial coordinate *r*, and ρ is the density of the medium. Relation between stress and strain for anisotropic layer is

$$\sigma_{r\theta}{}^{(2)} = N\left(\partial v/\partial r - v/r\right), \qquad \sigma_{z\theta}{}^{(2)} = L(\partial v/\partial z)$$

where *N* and *L* are the rigidities of the medium along *r* and *z* directions respectively. By using above relation, Eq. (13) may reduced as:

$$\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial z} - \frac{v}{r^2}\right) - \frac{1}{N}\frac{\partial}{\partial z}\left(G\frac{\partial v}{\partial z}\right) = \frac{\rho}{N}\frac{\partial^2 v}{\partial t^2} \tag{14}$$

where $G_n = L_{n0} - P_{n0}/2,\ n = 1,2,3$. Since the middle layer does not contain any initial stress i.e. $P_{20} = 0$. Assuming the solution of Eq. (14) as $v = V(z) J_1(kr) e^{i\omega t}$ where ω is the frequency, Eq. (14) takes the form

$$\frac{d^2 V}{dz^2} + \frac{1}{G_n}\frac{dG_n}{dz}\frac{dV}{dz} + \frac{N}{G_n}\left(k^2 - \frac{\omega^2 \rho}{N}\right) V(z) = 0 \tag{15}$$

Assuming

$$V(z) = V(z)/\sqrt{G_n} \tag{16}$$

Eq. (15) can be written as:

$$\frac{d^2 V}{dz^2} - \left\{\frac{1}{2G_n}\frac{d^2 G_n}{dz^2} - \left(\frac{1}{2G_n}\frac{dL}{dz}\right)^2 + \frac{N}{G_n}\left(k^2 - \frac{\omega^2 \rho}{N}\right)\right\} V_1 = 0 \tag{17}$$

where $\omega = kc$, c the phase velocity of the torsional surface waves in initial stressed half space.

4.1 Solution of lower homogeneous half-space

Consider the hyperbolic variation in elastic moduli, density and initial stress with depth *z* as:

$$N=N_{30}\cosh^2(\gamma z),L=L_{30}\cosh^2(\gamma z),\rho=\rho_{30}\cosh^2(\gamma z),P=P_{30}\cosh^2(\gamma z) \tag{18}$$

where γ is inhomogeneity parameter. Using relation (18), Eq. (17) takes the form

$$\frac{d^2V}{dz^2}-\lambda_2^2V(z)=0 \tag{19}$$

where $\lambda_2^2=k^2\left\{\frac{\gamma^2}{k^2}+\frac{N_{30}}{G_2}\left(1-\frac{c^2}{c_2^2}\right)\right\}$. The solution of Eq. (19) is given by

$$V(z)=B_2e^{-\lambda_2 z} \tag{20}$$

Therefore the solution of Eq. (13) is

$$v=v_2=\frac{B_2e^{-\lambda_2 z}J_1(kr)e^{i\omega t}}{\sqrt{G_0}\cosh(\gamma z)} \tag{21}$$

4.2 Solution of upper half-space

Consider the quadratic variation in elastic moduli, density and initial stress with depth z as:

$$N=N_{10}(1+\alpha z)^2,L=L_{10}(1+\alpha z)^2,\rho=\rho_{10}(1+\alpha z)^2,P=P_{10}(1+\alpha z)^2 \tag{22}$$

where α is the inhomogeneity parameter in lower half-space. Using relation (22), Eq. (17) takes the form

$$\frac{d^2V}{dz^2}-\lambda_0^2V(z)=0 \tag{23}$$

where $\lambda_0^2=\frac{k^2N_{30}}{G_2}\left(1-\frac{c^2}{c_2^2}\right)$ and $c_0=\sqrt{N_{10}/\rho_{10}}$ is the shear wave velocity in the medium along the radial direction. Solution of Eq. (23) is given by

$$V(z)=A_0e^{-\lambda_0 z}+B_0e^{\lambda_0 z}$$

The magnitude of the surface wave propagating in the medium decays rapidly with increase in depth, so the solution of Eq. (21) may be taken as:

$$V(z)=B_0e^{\lambda_0 z} \tag{24}$$

where A_0 and B_0 are arbitrary constant. In view of Eqs.(16) and (24), the solution of Eq. (13) takes the form

$$v=v_0=\frac{B_0e^{\lambda_0 z}J_1(kr)e^{i\omega t}}{\sqrt{G_1}(1+\alpha z)} \tag{25}$$

where $v_0(z)$ represents the displacement in lower half-space.

5 PHASE VELOCITY EQUATIONS

Phase velocity equations will be obtained by using of above mentioned boundary conditions into Eqs. (12), (21) and (25), we have following equations as:

$$\frac{e^{-\lambda_0 H}}{\sqrt{G_0}(1-\alpha H)} B_0 = \frac{1}{\sqrt{L_{20}}}\left(A_1 e^{-i\lambda_1 H} + B_1 e^{+i\lambda_1 H}\right) \tag{26}$$

$$\frac{L_{10}}{\sqrt{G_0}}\left\{\left(1-\alpha H\right)\lambda_0 - \alpha\right\} e^{-\lambda_0 H} B_0 = \sqrt{L_{20}} e^{-H/\beta}\left(-i\lambda_1 e^{i\lambda_1 H} A_1 + i\lambda_1 e^{-i\lambda_1 H} B_1\right) \tag{27}$$

$$\frac{1}{\sqrt{L_{20}}}\left(A_1 + B_1\right) = \frac{B_2}{\sqrt{G_2}} \tag{28}$$

$$\sqrt{L_{20}}\left(-i\lambda_1 A_1 + i\lambda_1 B_1\right) = -\frac{\lambda_2 L_{30}}{\sqrt{G_2}} B_2 \tag{29}$$

6 DISPERSION RELATION

Eliminating all the arbitrary constants from the phase velocity equations, we have

$$\tan(\lambda_1 H) = \frac{\left[\dfrac{L_{30}\lambda_1\lambda_2 e^{-\beta H}}{\left(1-\alpha H\right)} + L_{10}\lambda_1\left\{\left(1-\alpha H\right)\lambda_0 - \alpha\right\}\right]}{\left[\dfrac{L_{20}\lambda_1^2 e^{-\beta H}}{\left(1-\alpha H\right)} - \dfrac{L_{10}L_{30}\lambda_2\left\{\left(1-\alpha H\right)\lambda_0 - \alpha\right\}}{L_{20}}\right]} \tag{30}$$

Eq.(30) represents the dispersion relation of the torsional surface waves in the assumed heterogeneity model. Obtained dispersion relation does not coincide with the classical dispersion relation of Love waves in the presences of anisotropic, heterogeneity, initial stress. Now following particular cases are considered to reduce Eq. (30) into classical form of dispersion relation.

6.1 Case I

When $\alpha \to 0, N_{10} = L_{10}$ and $P_{10} = 0$ i.e. the upper half-space has constant rigidity and density, the Eq. (30) takes the form as:

$$\tan(\lambda_1 H) = \frac{\left[L_{30}\lambda_1\lambda_2 e^{-\beta H} + L_{10}\lambda_1 m_0\right]}{\left[L_{20}\lambda_1^2 e^{-\beta H} - \dfrac{L_{10}L_{30}\lambda_2 m_0}{L_{20}}\right]} \tag{31}$$

where $m_0^2 = k^2\left(1 - c^2/c_0^2\right)$ and the above Eq. (31) represents the dispersion relation of torsional surface waves when upper half-space is homogeneous.

6.2 Case II

If $\beta \to 0, N_{20} = L_{20}$ and $P_{20} = 0$, then Eq. (30) reduces to

$$\tan(m_1 H) = \frac{\left[\frac{L_{30} m_1 \lambda_2}{(1-\alpha H)} + L_{10} m_1 \{(1-\alpha H)\lambda_0 - \alpha\}\right]}{\left[\frac{L_{20} m_1^2}{(1-\alpha H)} - \frac{L_{10} L_{30} \lambda_2 \{(1-\alpha H)\lambda_0 - \alpha\}}{L_{20}}\right]} \tag{32}$$

where $m_1^2 = k^2 \left(c^2 / c_1^2 - 1\right)$ and the above Eq. (32) represents the dispersion relation of torsional surface wave in an inhomogeneous anisotropic layer.

6.3 Case III

Setting $\gamma \to 0, N_{30} = L_{30}$ and $P_{30} = 0$ in the Eq. (30), we get

$$\tan(\lambda_1 H) = \frac{\left[\frac{L_{30} \lambda_1 m_2 e^{-\beta H}}{(1-\alpha H)} + L_{10} \lambda_1 \{(1-\alpha H)\lambda_0 - \alpha\}\right]}{\left[\frac{L_{20} \lambda_1^2 e^{-\beta H}}{(1-\alpha H)} - \frac{L_{10} L_{30} m_2 \{(1-\alpha H)\lambda_0 - \alpha\}}{L_{20}}\right]} \tag{33}$$

where $m_2^2 = k^2 \left(1 - c^2 / c_2^2\right)$. In this case, Eq. (30) represents the dispersion relation of torsional surface wave for constant rigidity and density in the lower half-space.

6.4 Case IV

Considering $\alpha \to 0, \beta \to 0, \gamma \to 0, P_{10} = P_{20} = P_{30} = 0, N_{10} = L_{10}, N_{20} = L_{20} = \mu_1, N_{30} = L_{30} = \mu_2$ and neglecting the density (L_{10}) of upper half-space in Eq. (30) , we get

$$\tan\left\{kH\sqrt{c^2 / c_1^2 - 1}\right\} = \frac{\mu_2}{\mu_1} \frac{\sqrt{1 - c^2 / c_2^2}}{\sqrt{c^2 / c_1^2 - 1}} \tag{34}$$

which is the classical dispersion relation of the torsional surface wave in an heterogeneity model. Eq. (34) is a well known classical dispersion equation of Love wave (Love [13]). In this case the torsional wave mode changes into the Love wave mode and this conversion shows that such kind heterogeneity model exists in the Earth and allows the torsional surface waves to propagate.

7 NUMERICAL COMPUTATION AND DISCUSSIONS

The phase velocity c / c_1 of the torsional wave in an inhomogeneous anisotropic layer has been calculated numerically from the Eq.(30). The graphical representation shows the phase velocity relation for the different values of $\alpha / k, \gamma / k, \beta / k, P_{10} / 2L_{10}$ and $P_{30} / 2L_{30}$ with the fixed value of $P_{10} / L_{10} = 2, P_{20} / L_{20} = 0.7, P_{30} / L_{30} = 2, c_1 / c_2 = 0.2 = c_1 / c_0$, $L_{30} / L_{20} = 1.8$ and $L_{10} / L_{20} = 0.25$ for all the curves.

Fig. (2) to Fig. (6) have been plotted for non-dimensional wave number kH versus dimensionless phase velocity c/c_1. Fig. (2) to Fig. (4) represent the effect of inhomogeneity parameters on the phase velocity of torsional wave as well as Fig. (5) and Fig. (6) show the effect of initial stresses on the propagation of torsional waves. For graphical representation, MATLAB software has been used to generalize the results. It is observed that there is a significant effect of heterogeneity and initial stresses on the propagation of the torsional wave in an inhomogeneous anisotropic layer. It is found that, the phase velocity is more in the case of quadratic inhomogeneity(upper half-space) rather than hyperbolic (lower half-space).

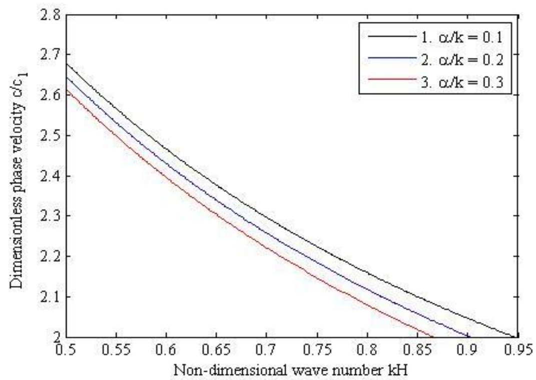

Fig.2
Dimensionless phase velocity c/c_1 versus non-dimensional wave number kH for the different values of $\alpha/k = 0.1, 0.2, 0.3$ with the fixed value of $P_{10}/2L_{10} = P_{30}/2L_{30} = 0.2, \gamma/k = 0.2$, and $\beta/k = 0.2$.

Fig. 2 represents the effect of inhomogeneity parameter α/k on the phase velocity of the torsional wave. In this figure, all curves have been plotted by taking $\alpha/k = 0.1, 0.2, 0.3$ and the constant value of other parameters have been taken as $P_{10}/2L_{10} = P_{30}/2L_{30} = 0.2, \gamma/k = 0.2$ and $\beta/k = 0.2$. The phase velocity of the torsional wave decreases rapidly as the value of α/k increases. It is also observed that the phase velocity c/c_1 decreases with the increases in wave number kH. The phase velocity of the torsional wave has been affected in sandwiched medium.

Fig. 3 depicts the effect of inhomogeneity parameter γ/k on the propagation of torsional wave. The value of γ/k for the curve 1, curve 2, curve 3 and curve 4 has been taken as 0.2, 0.4, 0.6 and 0.8 respectively and other parameters remains constant. In this figure the inhomogeneity parameter γ/k has the visible effect on the phase velocity of the torsional wave for low frequency and curves are shifted closer to each other at higher magnitude, that is, inhomogeneity parameter γ/k come into effect on the phase velocity of the torsional wave for low magnitude.

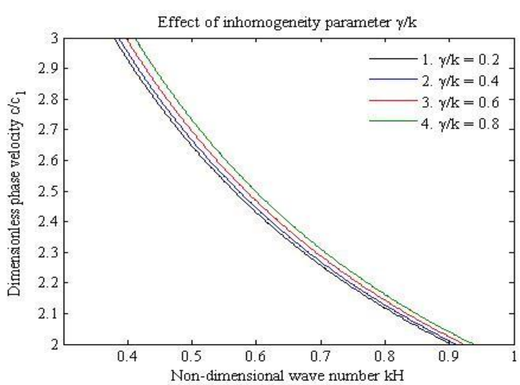

Fig.3
Dimensionless phase velocity c/c_1 versus non-dimensional wave number kH for the different values of $\gamma/k = 0.2, 0.4, 0.6, 0.8$ with the fixed value of $P_{10}/2L_{10} = P_{30}/2L_{30} = 0.2, \alpha/k = 0.2$ and $\beta/k = 0.2$.

From Fig. 2 and Fig. 3 it may be concluded that the phase velocity c/c_1 decreases with increases in wave number kH. The inhomogeneity parameter α/k affected the phase velocity of a torsional wave for the higher magnitude as shown in Fig. 2, whereas, the inhomogeneity parameter γ/k has negligible effect on the wave propagation after a certain point (0.7, 2.3). From both figures it is concluded that the heterogeneity parameter α/k in upper half-space is more effective than the heterogeneity parameter γ/k of lower half-space.

Fig. 4 gives the change of phase velocity for the different value of the inhomogeneity parameter $\beta/2k$ of anisotropic layer with compressive initial stresses and heterogeneity parameters (α/k and γ/k) and the fixed parameters have been taken as $P_{10}/2L_{10}=0.2, P_{30}/2L_{30}=0.2, \alpha/k=0.2=\gamma/k$. The value of $\beta/2k$ has been taken as 0.1, 0.2, 0.3 and 0.4 for curve 1, curve 2, curve 3 and curve 4 respectively. It may conclude that the phase velocity c/c_1 increases as the value of $\beta/2k$ increases for the wave number $kH<1$ and the effect of the inhomogeneity parameter $\beta/2k$ is vanished for certain wave number $kH>1$. It is also observed that, the phase velocity increases rapidly for the inhomogeneity parameter $\beta/2k>0.3$ as shown in curve 4.

Fig. 5 depicts the variation of phase velocity c/c_1 with respect to the wave number kH for different value of initial stress $P_{10}/2L_{10}$ associated with upper half-space. The value of initial stress $P_{10}/2L_{10}$ for curve 1, curve 2 and curve 3 is 0.2, 0.4 and 0.6, and the value of other parameters is $P_{30}/2L_{30}=0.2, \alpha/k=0.2=\gamma/k, \beta/2k=0.2$. The higher magnitude in initial stress $P_{10}/2L_{10}$ is more effective on the phase velocity of torsional wave and the phase velocity c/c_1 of a torsional wave increases as the value of $P_{10}/2L_{10}$ increases.

Fig. 6 shows the effect of initial stress $P_{30}/2L_{30}$ associated with lower half-space on the propagation of torsional wave in particular range. The value of initial stress $P_{30}/2L_{30}$ has been taken as 0.4, 0.6 and 0.8 for curve 1 to curve 3. It is observed that, the phase velocity increases with increases in initial stress $P_{30}/2L_{30}$ with respect to wave number kH.

The study on seismic waves gives important information about the layered Earth structure and has been used to determine the epicenter of earthquake. Seismologists are able to learn about Earth's internal structure by measuring the arrival of seismic waves at stations around the world because these waves travel at different speeds through different materials. Knowing how fast these waves travel through the Earth, seismologists can calculate the time when the earthquake occurred and its location by comparing the times when shaking was recorded at several stations. If a wave will arrive late, it would pass through a hot, soft part of the Earth. This study shows that phase velocity dispersion curve is affected by inhomogeneity and since the Earth's crust and mantle are non-homogeneous in nature, the findings may play a vital role in understanding the cause of damage due to earthquakes.

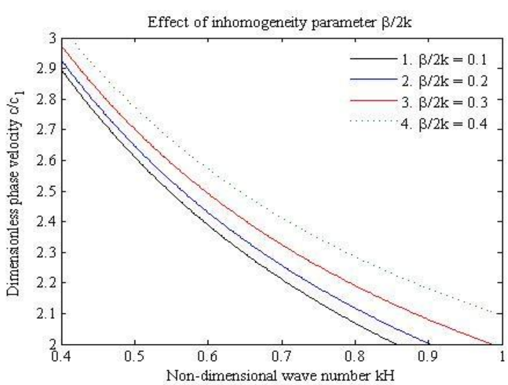

Fig.4
Dimensionless phase velocity c/c_1 versus non-dimensional wave number kH for the different values of $\beta/2k=0.1,0.2,0.3,0.4$ with the fixed value of $P_{10}/2L_{10}=P_{30}/2L_{30}=0.2, \alpha/k=0.2$, and $\gamma/k=0.2$.

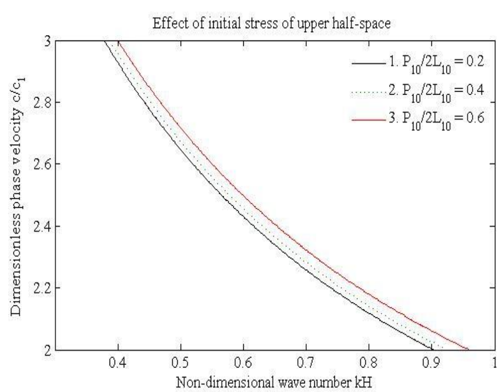

Fig.5
Dimensionless phase velocity c/c_1 versus non-dimensional wave number kH for the different values of $P_{10}/2L_{10}=0.2,0.4,0.6$ with the fixed value of $P_{30}/2L_{30}=0.2, \alpha/k=0.2, \beta/2k=0.2$, and $\gamma/k=0.2$.

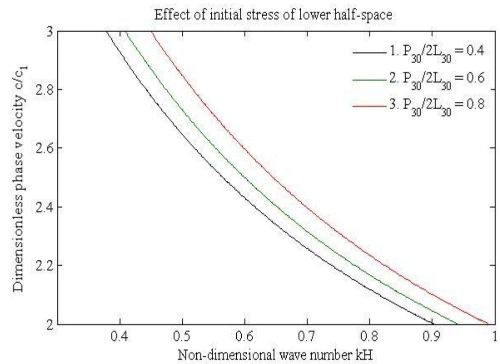

Fig.6
Dimensionless phase velocity c/c_1 versus non-dimensional wave number kH for the different values of $P_{30}/2L_{30} = 0.2, 0.4, 0.6$ with the fixed value of $P_{10}/2L_{10} = 0.2, \alpha/k = 0.2, \beta/2k = 0.2$ and $\gamma/k = 0.2$.

8 CONCLUSIONS

In this study the propagation of torsional surface waves in an inhomogeneous anisotropic layer lying between two heterogeneous half-spaces has been investigated analytically. We conclude that the geometry may allow the propagation of torsional surface wave in three different mediums(as shown in Fig. 1). Some special cases of interest have been deduced from the generalized dispersion relation as:

Case I described the dispersion relation of torsional surface waves in anisotropic layer and inhomogeneous semi-infinite medium(lower half-space). In this case, the phase velocity affected by the inhomogeneity parameters $\beta/2k = 0.2, \gamma/k$ and initial stress $P_{30}/2L_{30}$ as showing in Fig. 4, Fig. 3 and Fig. 6 respectively.

Case II represents the dispersion relation of the torsional wave in the absence of inhomogeneity parameter $\beta(\beta \to 0)$ of anisotropic layer. In other words, when the layer is taken as homogeneous medium. The effect of inhomogeneity parameters $\alpha/k, \gamma/k$ and initial streses $P_{10}/2L_{10}, P_{30}/2L_{30}$ on the propagation of the torsional wave has been demonstrated in the Fig. 2, Fig. 3, Fig. 5 and Fig. 6 and discussed respectively.

Case III discussed the relation of inhomogeneity parameters(α/k and $\beta/2k$) and initial stress $P_{10}/2L_{10}$ on the phase velocity c/c_1 of the torsional wave.

Case IV, in this case, when the initial stresses, inhomogeneity and rigidity L_{10} of the upper half-space are negligible, the dispersion equation coincides with the well-known classical equation of Love wave which is the validation of the problem.

From the above figures we may conclude that:

Dimensionless phase velocity c/c_1 of torsional surface wave decreases with increases of non-dimensional wave number kH with visible effect of initial stresses and heterogeneity of the mediums.

The phase velocity of the torsional wave increases with increases in heterogeneity parameters $\beta/2k$ and γ/k , whereas, the phase velocity decreases as the value of α/k increases.

The phase velocity of the torsional wave increases as the value of $P_{10}/2L_{10}$ and $P_{30}/2L_{30}$ increases.

It is found that, the phase velocity of torsional wave is more in the case of quadratic inhomogeneity rather than hyperbolic and exponential variation in rigidities and densities having the simultaneous effect on the phase velocity of the torsional wave. The obtained results are useful to find the location of earthquakes as well as their energy, mechanism etc. and may provide valuable information about the selection of proper structural materials for civil construction.

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