Vibration Analysis of FG Nanoplate Based on Third-Order Shear Deformation Theory (TSDT) and Nonlocal Elasticity

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ABSTRACT

In present study, the third-order shear deformation theory has been developed to investigate vibration analysis of FG Nano-plates based on Eringen nonlocal elasticity theory. The materials distribution regarding to the thickness of Nano-plate has been considered based on two different models of power function and exponential function. All equations governing on the vibration of FG Nano-plate have been derived from Hamilton’s principle. It has been also obtained the analytical solution for natural frequencies and corresponding mode shapes of simply supported FG Nano-plates. In addition, the general form of stiffness and mass matrix elements has been expressed based on this theory. The effect of different parameters such as power and exponential indexes of targeted function, nonlocal parameter of Nano-plate, aspect ratio and thickness to length ratio of Nano-plate on non-dimensional natural frequencies of free vibration responses have been investigated. The obtained analytical results show an excellent agreement with other available solutions of previous studies. The formulation and analytical results obtained from proposed method can be used as a benchmark for further studies to develop this area of research.

Keywords: Nano-plate; Functionally graded material (FGM); Nonlocal elasticity; Third order of shear deformation theory (TSDT); Natural frequency.

1 INTRODUCTION

IN recent years, the Nano-scale electromechanical systems (NEMS) such as high frequency Nano-actuators, Nano-sensors, Nano-super capacitors and Nano-semiconductor have been paid much attentions due to developments of engineering sciences. In the other sides, FG materials are a class of non-homogeneous materials obtained from a combination of the two materials to create a combination with specific functions. In FG materials, the possibility of delamination is decreased due to the stress concentration at the interface, with gradual change in volume fraction of compounds compere to their sudden change in multilayer composite materials [1]. Generally, Power law functions [2, 3] and exponential functions [4, 5] are used to describe the changes in properties of FG material. An important part of the studies have been conducted on the behavior of bending, vibration and buckling of one-dimensional nanostructures emphasized on nonlocal elasticity theory (Aydogdu [6], Civalk and Demir [7], Reddy [8, 10], Reddy and Pang [9], Roque et al [11] and Wang et al [12]). These nanostructures include Nano-beams, Nano-rods and carbonic nanotubes. In recent years, the application of FG materials has been highly developed in Nano-scale devices and systems such as thin films [13, 14 and 15], atomic force microscopy [16] and

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bio-mass sensor applications [17]. Salehipour et al have analytically investigated the free vibration FG Micro/Nano-plates with different combination of plain bearings and fixed bearings and free boundary conditions using Eringen nonlocal elasticity theory [18]. Nami and Janghorban have conducted a study on resonance behavior of rectangular FG Nano-plates with plain bearing boundary condition [19]. They utilized from theories independent from nonlocal elasticity scale and strain gradient. Salehipour et al developed exact Solution of the free vibration to FG Nano-plates using three-dimensional theories of elasticity [20, 21]. Natarajan et al have also analyzed vibration behavior of FG Nano-plates using first-order shear deformation theory [22].

In present study, the third-order shear deformation theory has been developed to investigate vibration analysis of FG Nano-plates based on Eringen nonlocal elasticity theory. All equations governing on the vibration of FG Nano-plate have been derived from Hamilton’s principle. It has been also obtained the analytical solution for natural frequencies and corresponding mode shapes of simply supported FG Nano-plates. Finally, the effect of different parameters such as power and exponential indexes of targeted function, nonlocal parameter of Nano-plate, aspect ratio and thickness to length ratio of Nano-plate on non-dimensional natural frequencies of free vibration responses have been investigated.

2 MODELING AND DESCRIPTION OF RELATIONS GOVERNING ON THE PROBLEM

2.1 A brief history of nonlocal elasticity theory

The nonlocal elasticity theory was firstly introduced by Eringen to take into account the effect of small scale parameter in modeling continuum mechanics in non-classical problems [23]. In nonlocal theory despite of classical elasticity theory, the elasticity is modeled at a single point of continuous physical model which depends on the strain of all its parts. In the other words, strain in a single point depends on the elasticity and its partial derivatives in mentioned point.

Eringen has expressed the differential equation of this theory in a way that Non-local elasticity tensor is signed with $\sigma_{ij}$ and local elasticity tensor with $t_{ij}$.

$$\left(1-\mu N^2\right)\sigma_{ij} = t_{ij} \quad (1)$$

2.2 The model of functionally graded material

2.2.1 The model of exponential functionally graded material

The amount and how a change in properties of the ceramic material on the $z$ axis is shown with exponential index of FG material i.e. $\zeta$ parameter. Following exponential functions are used to describe the properties distribution in the materials:

$$P(z) = P_e e^{\zeta \left(\frac{z}{h} - \frac{1}{2}\right)} \quad (2)$$

where, $z$ is the thickness of the plate and $P(z)$ indicate a general property of material such as Young's modulus. The coefficients $P_e$ and $P_a$ are high-level properties (full ceramic) and low-level properties (full metal) of the material, respectively.

2.2.2 The model of exponential functionally graded material

The amount of volume distribution fraction of ceramic material on the $z$ axis is shown with power index of FG material i.e. $\zeta$ and is defined using following relation:

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^\zeta \quad (3)$$
where $V_c$ is the ceramic material parameter on its distribution profile. After defining volume distribution fraction for this material, it should be noted that the numerical parameters or power index of FG material is a positive number; in a way that if $\zeta$ equals to zero, then the beam is quite ceramic and the metallicity increased with its increasing toward infinity rate. Different properties of material vary along the thickness to form:

$$P(z) = PV_c + P_mV_m$$

(4)

2.3 Equilibrium equations with third-order shear deformation theory (TSDT) and nonlocal elasticity

Fig. 1 represents the geometric model of rectangular plate system with sides $A$ and $B$ and thickness $h$ made up of FG materials.

![Fig.1](image)

Geometric model of system.

2.3.1 The relations of third-order shear deformation theory

According to the Reddy shear theory for thick plates, the deformation field in plate can be rewritten as follow:

$$u(x, y, z) = u_0(x, y) + z\phi_x - \frac{4z^3}{3h^2}\left(\phi_x + \frac{\partial w_x}{\partial x}\right)$$

$$v(x, y, z) = v_0(x, y) + z\phi_y - \frac{4z^3}{3h^2}\left(\phi_y + \frac{\partial w_y}{\partial y}\right)$$

(5)

$$w(x, y, z) = w_o(x, y)$$

In addition, the strain-displacement relations in third-order shear deformation theory are according to the following equations:

$$\begin{bmatrix}
E_{xx} \\
E_{yy} \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
E_{xx}^{(0)} \\
E_{yy}^{(0)} \\
\gamma_{xy}^{(0)}
\end{bmatrix} + z\begin{bmatrix}
E_{xx}^{(1)} \\
E_{yy}^{(1)} \\
\gamma_{xy}^{(1)}
\end{bmatrix} + z^2\begin{bmatrix}
E_{xx}^{(2)} \\
E_{yy}^{(2)} \\
\gamma_{xy}^{(2)}
\end{bmatrix}$$

(6)
\[
\begin{align*}
\left\{ \begin{array}{c}
\varepsilon_x^{(1)} \\
\varepsilon_y^{(1)} \\
\gamma_{xy}^{(1)}
\end{array} \right\} &= -c_1 \left\{ \begin{array}{c}
\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w_0}{\partial x \partial y}
\end{array} \right\} \\
\left\{ \begin{array}{c}
\varepsilon_x^{(3)} \\
\varepsilon_y^{(3)} \\
\gamma_{xy}^{(3)}
\end{array} \right\} &= -c_1 \left\{ \begin{array}{c}
\frac{\partial \phi_x}{\partial x} \\
\frac{\partial \phi_y}{\partial y} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{array} \right\} \\
\left\{ \begin{array}{c}
\gamma_{xy}^{(9)} \\
\gamma_{xy}^{(9)}
\end{array} \right\} &= \left\{ \begin{array}{c}
\phi_x + \frac{\partial w_0}{\partial x} \\
\phi_y + \frac{\partial w_0}{\partial y}
\end{array} \right\} \\
\left\{ \begin{array}{c}
\gamma_{xy}^{(2)} \\
\gamma_{xy}^{(2)}
\end{array} \right\} &= -c_2 \left\{ \begin{array}{c}
\phi_x + \frac{\partial w_0}{\partial x} \\
\phi_y + \frac{\partial w_0}{\partial y}
\end{array} \right\}
\end{align*}
\]  

where \( c_1 = \frac{4}{3h} \), \( c_2 = 3c_1 \).

Therefore, Hooke's law can be written in the following form:

\[
\begin{align*}
\left\{ \begin{array}{c}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{array} \right\} &= E(z) \left[ \begin{array}{ccc}
\frac{1}{1-\nu^2} & \frac{\nu}{1-\nu^2} & 0 \\
\frac{\nu}{1-\nu^2} & \frac{1}{1-\nu^2} & 0 \\
0 & 0 & \frac{1}{2(1+\nu)}
\end{array} \right] \left\{ \begin{array}{c}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{array} \right\} \\
\left\{ \begin{array}{c}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xx}
\end{array} \right\} &= E(z) \left[ \begin{array}{ccc}
\frac{1-\nu}{2} & \frac{\nu}{2} & 0 \\
\frac{1-\nu}{2} & \frac{\nu}{2} & 0 \\
\frac{1-\nu}{2} & \frac{\nu}{2} & 0
\end{array} \right] \left\{ \begin{array}{c}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{array} \right\}
\end{align*}
\]  

The final strain-displacement relations for the plate can be written as following form:

\[
\begin{align*}
\left\{ \begin{array}{c}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{array} \right\} &= E(z) \left[ \begin{array}{ccc}
\frac{\partial u}{\partial x} + \frac{\partial \phi_x}{\partial x} & \frac{\partial \phi_y}{\partial y} & \frac{\partial w_0}{\partial x} \\
\frac{\partial v}{\partial y} & \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} & \frac{\partial w_0}{\partial y} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} & \frac{\partial \phi_y}{\partial y} + \frac{\partial w_0}{\partial x} & \frac{\partial w_0}{\partial y}
\end{array} \right] \left\{ \begin{array}{c}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{array} \right\} \\
&\quad + z \left[ \begin{array}{ccc}
\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial x} & \frac{\partial \phi_y}{\partial y} & \frac{\partial w_0}{\partial x} \\
\frac{\partial \phi_x}{\partial y} & \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} & \frac{\partial w_0}{\partial y} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} & \frac{\partial \phi_y}{\partial y} + \frac{\partial w_0}{\partial x} & \frac{\partial w_0}{\partial y}
\end{array} \right] \left\{ \begin{array}{c}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{array} \right\}
\end{align*}
\]  

The resultants elasticity in nonlocal theory for Nano-plate is:

\[
\begin{align*}
\left\{ \begin{array}{c}
N_{xx}^{nl} \\
N_{yy}^{nl} \\
N_{xy}^{nl}
\end{array} \right\} + \left\{ \begin{array}{c}
M_{xx}^{nl} \\
M_{yy}^{nl} \\
M_{xy}^{nl}
\end{array} \right\} = \left\{ \begin{array}{c}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{array} \right\} + \left\{ \begin{array}{c}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{array} \right\} - \left\{ \begin{array}{c}
P_{xx} \\
P_{yy} \\
P_{xy}
\end{array} \right\}
\end{align*}
\]
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which, the equations related to \(N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}, P_{xx}, P_{yy}\) have been represented in the appendix.

2.3.2 Equilibrium equations of the system

The virtual work method and Hamilton's principle have been used to achieve motion equation of the system.

\[
\int_0^{b/2} \int_{-h/2}^{h/2} \left( \sum_{xx} \delta_e \delta_e + \sum_{yy} \delta_e \delta_e + 2 \sum_{xy} \delta_e \delta_e + 2 \sum_{xy} \delta_e \delta_e - 0.5 \rho(z) \left( \ddot{u}_{xy}^2 + \ddot{v}_{xy}^2 + \ddot{w}_{xy}^2 \right) \right) \, du \, dv \, dw \\
\frac{d \dot{u}^2}{dt} = 0
\]

(11)

According to the Hamilton's principle, the equilibrium equations of system in nonlocal space can be expressed as following five forms:

\[
\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{yy}}{\partial y} - (1 - \mu N^2) \left( I_o u_{xx} + J_1 \phi_x - c_1 I_3 \frac{\partial w_0}{\partial x} \right) = 0
\]

(12)

\[
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{xy}}{\partial y} - (1 - \mu N^2) \left( I_o u_{xy} + J_1 \phi_x - c_1 I_3 \frac{\partial w_0}{\partial y} \right) = 0
\]

(13)

\[
\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - \ddot{Q}_{xx}^m - (1 - \mu N^2) \left( J_1 I_{xx} + K_2 \phi_x - c_1 I_3 \frac{\partial w_0}{\partial x} \right) = 0
\]

(14)

\[
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - \ddot{Q}_{xy}^m - (1 - \mu N^2) \left( J_1 I_{xy} + K_2 \phi_x - c_1 I_3 \frac{\partial w_0}{\partial y} \right) = 0
\]

(15)

\[
\frac{\partial Q_{xx}^m}{\partial x} + \frac{\partial Q_{xy}^m}{\partial y} + \frac{\partial}{\partial x} \left( N_{xx}^m \frac{\partial w_0}{\partial x} + N_{yy}^m \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy}^m \frac{\partial w_0}{\partial x} + N_{yy}^m \frac{\partial w_0}{\partial y} \right) + c_1 \left( \frac{\partial^2 P_{xx}^m}{\partial x^2} + 2 \frac{\partial^2 P_{xy}^m}{\partial x \partial y} + \frac{\partial^2 P_{yy}^m}{\partial y^2} \right)
\]

(16)

In above relations we have:

\[
\ddot{Q}_{xx}^m = M_{xx}^m - c_1 P_{xx}^m \quad (\alpha, \beta = 1, 2, 6)
\]

\[
\ddot{Q}_{xy}^m = Q_{xy}^m - c_1 R_{xy}^m \quad (\alpha = 4, 5)
\]

In addition, the phrases related to the high order of FG Nano-plate have been used in the equilibrium equations of system are in following form:

\[
I_i = \int_{-h/2}^{b/2} \rho(z) \, dz \quad (i = 0, 1, 2, ..., 6) \quad J_1 = I_1 - c_1 I_{1,2} \quad K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6
\]

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3 THE ANALYTICAL SOLUTION OF EQUATION

3.1 The boundary conditions governing on the problem

According to the model, the boundary conditions system of the plate has been considered in the form of four sides simply supported. The geometrical and mechanical conditions at the borders of the plate can be shown using following relations:

\[ u_o (x,0,t) = 0, u_o (x,b,t) = 0 \]
\[ v_0 (0,y,t) = 0, v_0 (a,y,t) = 0 \]
\[ \phi_x (x,0,t) = 0, \phi_x (x,b,t) = 0 \]
\[ \phi_y (0,y,t) = 0, \phi_y (a,y,t) = 0 \]
\[ w_o (x,0,t) = 0, w_o (x,b,t) = 0 \]
\[ w_0 (0,y,t) = 0, w_0 (a,y,t) = 0 \]
\[ N_\gamma (x,0,t) = 0, N_\gamma (x,b,t) = 0 \]
\[ N_\alpha (0,y,t) = 0, N_\alpha (a,y,t) = 0 \]
\[ M_\gamma (x,0,t) = 0, M_\gamma (x,b,t) = 0 \]
\[ M_\alpha (0,y,t) = 0, M_\alpha (a,y,t) = 0 \]

Therefore, following series which satisfy above conditions are considered as linear motion and angular functions of the plate.

\[ u_o (x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{mn} (t) \cos (\alpha x) \sin (\beta y) \]
\[ v_0 (x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{mn} (t) \sin (\alpha x) \cos (\beta y) \]
\[ w_o (x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn} (t) \sin (\alpha x) \sin (\beta y) \]
\[ \phi_x (x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \chi_{mn} (t) \cos (\alpha x) \sin (\beta y) \]
\[ \phi_y (x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{mn} (t) \sin (\alpha x) \cos (\beta y) \]

where, \( \beta = \frac{n \pi}{b}, \alpha = \frac{m \pi}{a} \) and \( U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn} \) are unknown parameters.

3.2 Analyzing free vibrations of the system

To analyze of system free vibration frequencies, it has been assumed that the time response is in harmonic form and it is possible to consider the time displacement vector \( \tilde{\Delta} = [U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}]^T \) in the form of \( e^{i \omega t} \tilde{\Delta} = \tilde{\Delta}_0 \). The matrix form of vibration’s equation of the system is as follow:

\[ ([K] - \omega^2 [M]) \tilde{\Delta} = 0 \] (21)

\([k]\) and \([m]\) are thickness and mass matrixes of the system, respectively are in square form with five rows and columns based on the third-order shear deformation theory. The frequency of free vibrations of the system has been signed with \( \omega \) which is based on radians per second. The determinant of the coefficient’s matrix should be zero to
achieve unique solutions form the system and the natural frequencies are the roots of the characteristic equation of system.

4 NUMERICAL RESULTS AND DISCUSSION OF THEM

4.1 Validation

To validate this analytical solution, its results have been comprised with previous studies. Since, no study have been yet conducted on the free vibration of thick FG Nano-plate (with power or exponential) based on the third-order shear deformation theory (TSDT) and nonlocal elasticity, so the results obtained from this analysis have been comprised from following references, respectively.


The study of Natarajan et al [22] and Zare et al [27] on investigation of free vibration of thin FG Nano-plate (with power function) using classical theory of plates and nonlocal elasticity.

The study of Salehipour et al [21] on investigation of free vibration of FG Nano-plate (with exponential function) using nonlocal elasticity theory, first order shear deformation theory and three-dimensional elasticity theory.

4.1.1 Comprising the results with references [25] and [26]

The following results were obtained for homogeneous Nano-plate with properties of $E = 300 GPa, a = 10nm$ and dimensionless frequency in the form of $\Omega = \rho a^2 \sqrt{\frac{h}{D}}$.

The ratio of nonlocal frequency to local frequency (a number between zero and one) in different modes and for different dimensionless nonlocal parameters of $nl = \frac{\sqrt{\mu}}{a}$ has been expressed in Table 1.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Num</th>
<th>Refs</th>
<th>$nl=0$</th>
<th>$nl=0.2$</th>
<th>$nl=0.4$</th>
<th>$nl=0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>[26]</td>
<td>Present Study</td>
<td>0.7475</td>
<td>0.4904</td>
<td>0.3512</td>
<td></td>
</tr>
<tr>
<td>(1,2)</td>
<td>[26]</td>
<td>Present Study</td>
<td>0.7475</td>
<td>0.4904</td>
<td>0.3512</td>
<td></td>
</tr>
<tr>
<td>(1,3)</td>
<td>[26]</td>
<td>Present Study</td>
<td>0.4497</td>
<td>0.244</td>
<td>0.1655</td>
<td></td>
</tr>
</tbody>
</table>

Table 2 represents a comparison among dimensionless fundamental frequencies for a homogeneous rectangular FG Nano-plate with $a/b = 0.5$ for different dimensionless parameters as well as different ratios of height to length of Nano-plate in different references.

<table>
<thead>
<tr>
<th>$h/a$</th>
<th>(nm$^2$)$\mu$</th>
<th>Refs</th>
<th>$\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[25]</td>
<td>12.1157</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[26]</td>
<td>12.0675</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Present Study</td>
<td>12.0675</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[25]</td>
<td>11.4187</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[26]</td>
<td>11.3856</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Present Study</td>
<td>11.3856</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>[25]</td>
<td>9.9016</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[26]</td>
<td>9.8745</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Present Study</td>
<td>9.8746</td>
<td></td>
</tr>
</tbody>
</table>
4.1.2 Comprising the results with references [22] and [27]

In this reference, the dimensionless frequency of FG plates was obtained based on the classical theory and the theory of nonlocal which its relation is in the form of \( \Omega_1 = \frac{\rho_c}{E_c} \sqrt{\frac{\rho_n}{G}} \). The comparison was carried out on a Nano-plate with following properties. Other geometrical and mechanical parameters have been presented as follow:

\[ E_c = 348.43 \text{GPa}, \quad E_n = 201.04 \text{GPa}, \quad G_i = \frac{E_c}{2(1 + \nu)}, \quad \rho_c = 2370, \quad \rho_n = 8166, \quad \nu = 0.3, \quad a = 10\text{nm} \]

Firstly, Table 3. was provided to comprise dimensionless frequencies in different dimensional ratios of FG Nano-plate as well as its different nonlocal parameters by fixing exponential index of FG material, its thickness to length ratio equal to 0.05 and power index parameter equal to 5.

### Table 3

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>Nonlocal</th>
<th>Refs</th>
<th>mode1</th>
<th>mode2</th>
<th>mode3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[22]</td>
<td>0.0113</td>
<td>0.0278</td>
<td>0.0279</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>[27]</td>
<td>0.0114</td>
<td>0.0281</td>
<td>0.0281</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Present Study</td>
<td>0.01133</td>
<td>0.02794</td>
<td>0.02794</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>[22]</td>
<td>0.0085</td>
<td>0.0161</td>
<td>0.0162</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[27]</td>
<td>0.0085</td>
<td>0.0165</td>
<td>0.0165</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Present Study</td>
<td>0.00847</td>
<td>0.0162</td>
<td>0.0162</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>[22]</td>
<td>0.0279</td>
<td>0.044</td>
<td>0.0701</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[27]</td>
<td>0.0281</td>
<td>0.0443</td>
<td>0.0704</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Present Study</td>
<td>0.02794</td>
<td>0.0441</td>
<td>0.07014</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>[22]</td>
<td>0.0162</td>
<td>0.0216</td>
<td>0.0283</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[27]</td>
<td>0.0165</td>
<td>0.0218</td>
<td>0.0286</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Present Study</td>
<td>0.0162</td>
<td>0.02163</td>
<td>0.02833</td>
<td></td>
</tr>
</tbody>
</table>

The dimensionless frequencies were also in good agreement for another FG Nano-plate with following properties:

\[ E_c = 380 \text{GPa}, \quad E_n = 70 \text{GPa}, \quad \rho_c = 2702, \quad \rho_n = 3800, \quad \nu = 0.3 \]

By fixing all dimensions of Nano-plate, the dimensionless frequency changes of \( \Omega_i = \frac{a^2}{h} \sqrt{\frac{\rho_c}{E_c}} \), were comprised with nonlocal parameter, as well as power index of FG material for two theories in different modes, which its results have been represented in Table 4.

### Table 4

<table>
<thead>
<tr>
<th>Nonlocal</th>
<th>( \zeta )</th>
<th>Theory</th>
<th>(1,1)</th>
<th>(1,2)</th>
<th>(2,2)</th>
<th>(1,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>CPT [27]</td>
<td>5.81138</td>
<td>14.3717</td>
<td>22.8824</td>
<td>28.5292</td>
<td></td>
</tr>
<tr>
<td></td>
<td>TSDT</td>
<td>5.52018</td>
<td>13.6247</td>
<td>21.5513</td>
<td>26.6988</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>CPT [27]</td>
<td>4.62678</td>
<td>11.4209</td>
<td>18.1769</td>
<td>22.6604</td>
<td></td>
</tr>
<tr>
<td></td>
<td>TSDT</td>
<td>3.81261</td>
<td>9.41587</td>
<td>14.8886</td>
<td>18.4687</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>CPT [27]</td>
<td>3.87362</td>
<td>9.53273</td>
<td>15.1478</td>
<td>18.8672</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>CPT [27]</td>
<td>5.32051</td>
<td>11.8058</td>
<td>17.2163</td>
<td>20.4084</td>
<td></td>
</tr>
<tr>
<td></td>
<td>TSDT</td>
<td>5.04469</td>
<td>11.1488</td>
<td>16.0952</td>
<td>18.9407</td>
<td></td>
</tr>
<tr>
<td></td>
<td>TSDT</td>
<td>3.48421</td>
<td>7.70479</td>
<td>11.296</td>
<td>13.1021</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>CPT [27]</td>
<td>3.54899</td>
<td>7.83929</td>
<td>11.4151</td>
<td>13.5229</td>
<td></td>
</tr>
<tr>
<td></td>
<td>TSDT</td>
<td>2.52281</td>
<td>5.50527</td>
<td>7.97921</td>
<td>9.36521</td>
<td></td>
</tr>
</tbody>
</table>
4.1.3 Comprising the results with reference [21]

For an exponential FG material, the results of this frequency analysis were also comprised with three-dimensional theory of the plates, as well as with first order of shear deformation in two-dimensional theory of plates. Table 5. represents the dimensionless fundamental frequencies of FG Nano-plate's free vibration in the two primary modes.

Table 5
Comparison between dimensionless fundamental frequencies for FG plate with various power indexes and height to length ratio parameters by reference [21].

<table>
<thead>
<tr>
<th>Nonlocal</th>
<th>Exp. Index</th>
<th>$a/h$</th>
<th>Method</th>
<th>First</th>
<th>Second</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>3D</td>
<td>FSDT</td>
<td>5.2476</td>
<td>13.777</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>TSDT</td>
<td>5.2283</td>
<td>13.777</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>3D</td>
<td>FSDT</td>
<td>5.7054</td>
<td>27.554</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>TSDT</td>
<td>5.7043</td>
<td>27.554</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>3D</td>
<td>FSDT</td>
<td>5.1575</td>
<td>13.777</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>TSDT</td>
<td>5.1504</td>
<td>13.777</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>3D</td>
<td>FSDT</td>
<td>5.6142</td>
<td>27.554</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>TSDT</td>
<td>5.6089</td>
<td>27.554</td>
</tr>
<tr>
<td>0.3</td>
<td>2</td>
<td>3D</td>
<td>FSDT</td>
<td>4.7032</td>
<td>12.388</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>TSDT</td>
<td>4.7011</td>
<td>12.388</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>3D</td>
<td>FSDT</td>
<td>5.5436</td>
<td>26.772</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>TSDT</td>
<td>5.5426</td>
<td>26.772</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>3D</td>
<td>FSDT</td>
<td>4.6374</td>
<td>12.388</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>TSDT</td>
<td>4.6311</td>
<td>12.388</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>3D</td>
<td>FSDT</td>
<td>5.4524</td>
<td>26.772</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3D</td>
<td>TSDT</td>
<td>5.4499</td>
<td>26.772</td>
</tr>
</tbody>
</table>

In general, the correctness and accuracy of present study was validated and confirmed through analyzing and comprising its results with above mentioned references.

4.2 Parametric study of the results

In this section, the effect of fundamental and variable parameters of present study on the vibration’s frequency has been investigated.

4.2.1 The effect of FG material index and nonlocal parameter

In Fig. 2 by linear increasing of power index value of FG material in an unique nonlinear parameter, the dimensionless frequency of Nano-plate's vibrations is decreased in a nonlinear form (it has trended from dimensionless frequency of pure ceramic material toward dimensionless frequency of pure metal material); in a way whatever the power index of FG material is greater, the curve slope of changes is more decreased and the sensitivity of frequency to index changes is reduced.

In a fixed power index, whatever the value of nonlocal parameter is lower, the dimensionless frequency of base mode vibration is lower. Therefore, the blue curve which indicates the mode of local classical theory always estimates the Nano-plate frequency higher than its actual value. This reduction in frequency is more significant when the value of index is lower than 2.
In Fig. 3 according to the horizontal axis which represents the dimensionless parameter value of exponential index, two modes can be considered to the unique nonlocal parameter: in the first mode if the dimensionless index be between zero and one, then by increasing the value from very small amounts, the frequency trends toward a maximum value with a very sharp slope and in a nonlinear form while the exponential index is equal to one. The maximum value varies for any different nonlocal parameters of Nano-plate, but all of them occur in horizontal coordinate of 1. In the second mode for index values of higher than 1, the frequency of base mode is reduced with a slight slope by an increase in the exponential index. Similar to the power mode, nonlocal parameter increasing leads to a nonlinear decrease in dimensionless frequency.

5 CONCLUSIONS

In present study, the third-order shear deformation theory has been developed to investigate vibration analysis of FG Nano-plates based on Eringen nonlocal elasticity theory. The materials distribution regarding to the thickness of Nano-plate has been considered based on two different models of power function and exponential function. All equations governing on the vibration of FG Nano-plate have been derived from Hamilton’s principle. It has been also obtained the analytical solution for natural frequencies and corresponding mode shapes of simply supported FG Nano-plates. In addition, the general form of stiffness and mass matrices' elements has been expressed based on this theory. In addition, the general form of mass matrix elements and thickness of FG Nano-plate have been expressed according to this theory.

The effect of different parameters including power and exponential indexes of FG, nonlocal parameter of Nano-plate, dimensional ratio and thickness to length ratio of Nano-plate on the dimensionless frequency of free vibrations were investigated.

The results showed that an increase in nonlocal parameter, as well as increase in power or exponential indexes of FG Nano-plate leads to a decrease in structure stiffness of Nano-plate-based system which is detectable through reduction in the frequency of system's oscillations.

APPENDIX

The relations related to the elasticity resultants can be written in following form:
\[
\begin{pmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{pmatrix}
\begin{pmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{pmatrix}
\begin{pmatrix}
P_{xx} \\
P_{yy} \\
P_{xy}
\end{pmatrix}
= \int_\Delta \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{pmatrix} (1, z, z') \, dA
\] (22)

\[
\begin{pmatrix}
Q_{xx} \\
Q_{yy} \\
R_{xx} \\
R_{yy}
\end{pmatrix}
= \int_\Delta \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz}
\end{pmatrix} (z) \, dA
\]

The strains-based elasticity resultants are obtained by replacing related equations:

\[
\begin{pmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{pmatrix}
= \begin{pmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(1)} \\
\varepsilon_{yy}^{(1)} \\
\gamma_{xy}^{(1)}
\end{pmatrix}
+ \begin{pmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(2)} \\
\varepsilon_{yy}^{(2)} \\
\gamma_{xy}^{(2)}
\end{pmatrix}
+ \begin{pmatrix}
E_{11} & E_{12} & E_{16} \\
E_{12} & E_{22} & E_{26} \\
E_{16} & E_{26} & E_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(3)} \\
\varepsilon_{yy}^{(3)} \\
\gamma_{xy}^{(3)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{pmatrix}
= \begin{pmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(1)} \\
\varepsilon_{yy}^{(1)} \\
\gamma_{xy}^{(1)}
\end{pmatrix}
+ \begin{pmatrix}
F_{11} & F_{12} & F_{16} \\
F_{12} & F_{22} & F_{26} \\
F_{16} & F_{26} & F_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(2)} \\
\varepsilon_{yy}^{(2)} \\
\gamma_{xy}^{(2)}
\end{pmatrix}
+ \begin{pmatrix}
H_{11} & H_{12} & H_{16} \\
H_{12} & H_{22} & H_{26} \\
H_{16} & H_{26} & H_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(3)} \\
\varepsilon_{yy}^{(3)} \\
\gamma_{xy}^{(3)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
P_{xx} \\
P_{yy} \\
P_{xy}
\end{pmatrix}
= \begin{pmatrix}
E_{11} & E_{12} & E_{16} \\
E_{12} & E_{22} & E_{26} \\
E_{16} & E_{26} & E_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(1)} \\
\varepsilon_{yy}^{(1)} \\
\gamma_{xy}^{(1)}
\end{pmatrix}
+ \begin{pmatrix}
F_{11} & F_{12} & F_{16} \\
F_{12} & F_{22} & F_{26} \\
F_{16} & F_{26} & F_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(2)} \\
\varepsilon_{yy}^{(2)} \\
\gamma_{xy}^{(2)}
\end{pmatrix}
+ \begin{pmatrix}
H_{11} & H_{12} & H_{16} \\
H_{12} & H_{22} & H_{26} \\
H_{16} & H_{26} & H_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx}^{(3)} \\
\varepsilon_{yy}^{(3)} \\
\gamma_{xy}^{(3)}
\end{pmatrix}
\]

where, the high order tensile and flexural stiffness matrixes of FG plate in third-order shear deformation theory are defined in following forms:

\[
\begin{pmatrix}
A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}\end{pmatrix} = \begin{cases}
\frac{b^2}{h^2} & \text{if } i,j = 1,2,6. \\
0 & \text{otherwise}
\end{cases}
\]

(24)

REFERENCES


