Fundamental Solution in the Theory of Thermoelastic Diffusion Materials with Double Porosity

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ABSTRACT

The main purpose of present article is to find the fundamental solution of partial differential equations in the generalized theory of thermoelastic diffusion materials with double porosity in case of steady oscillations in terms of elementary functions.

Keywords : Thermoelastic; Diffusion; Double porosity; Steady oscillations.

1 INTRODUCTION

LORD and Shulman [1] established the theory of generalized thermoelasticity by modifying Fourier’s law of heat conduction. This theory overcomes the shortcomings of classical theory of thermoelasticity in which thermal waves propagate with infinite velocity. The transfer of mass of a substance from the high concentration regions to low concentration regions is known as diffusion. Nowacki [2-5] developed the classical thermoelasticity with mass diffusion. With the help of modified Fourier’s and Fick’s laws, Sherief et al. [6] established generalized theory of thermoelasticity with mass diffusion. Iesan [7] constructed the linear theory of thermelastic materials with single voids. Aouadi [8] developed a theory of thermoelastic diffusion materials with voids and derived various theorems. The double porosity model represents a double porous structure, one is macro porosity which is connected to pores and other is micro porosity which is connected to fissures. Barenblatt et al. [9] and Warren and Root [10] extended Darcy’s law to describe fluid flow through undeformable double porosity materials. Wilson and Aifantis [11] developed the theory for deformable materials with double porosity. Iesan and Quintanilla [12] derived a non-linear theory of thermoelastic solids with double porosity structure based upon Nunziato-Cowin theory of materials with voids. This theory was not based upon Darcy’s law. Kansal [13] established linear generalized theory of thermoelastic diffusion with double porosity. The construction of fundamental solution of a system of partial differential equations and establishment of their basic properties are required to study the boundary value problems by using potential method. The concept of fundamental solutions has great importance in many mathematical, mechanical, physical, and engineering applications. For example, the application of fundamental solutions to a recently developed area of boundary element methods has provided a concrete advantage in that an integral representation of the solution to a boundary value problem in terms of a fundamental solution can be solved more easily by numerical methods with respect to a differential equation with some specified boundary and initial conditions. Several authors [14-18] constructed the fundamental solutions in the theories of elasticity and thermoelasticity for materials with double porosity.

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In this paper, firstly the basic equations for homogeneous anisotropic generalized thermoelastic diffusion solid with double voids are considered. After reducing to the isotropic case and assuming the solutions in case of steady oscillations, the fundamental solutions of the governing equations are constructed in terms of elementary functions. Some basic properties of fundamental matrix are also discussed. Finally, some particular cases are obtained.

2 BASIC EQUATIONS

Let \( \mathbf{x} = (x_1, x_2, x_3) \) be the point of the Euclidean three-dimensional space \( \mathbb{E}^3, |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \), \( \mathbf{D}_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \) and let \( t \) denote the time variable. Following Kansal [13], the governing equations for an anisotropic homogeneous generalized thermoelastic diffusion solid with double porosity in the absence of body forces, heat and mass diffusive sources are

Constitutive relations:

\[ \sigma_{ij} = c_{ijkl}e_{kl} + p_{ij} \phi + \gamma_{ij} \psi - a_{ij} \theta - b_{ij} C \] (1)

\[ \Omega_{i} = q_{ij} \phi_{,j} + \alpha_{ij} \psi_{,j} \] (2)

\[ \chi_{i} = \alpha_{ij} \psi_{,j} + f_{ij} \psi_{,j} \] (3)

\[ \xi = -p_{ij} e_{ij} - d_{ij} \phi_{,j} - a_{ij} \psi_{,j} + \gamma_{ij} \theta_{,j} + vC \] (4)

\[ \zeta = -\gamma_{ij} e_{ij} - \alpha_{ij} \phi_{,j} - f_{ij} \psi_{,j} + \gamma_{ij} \theta_{,j} + mC \] (5)

\[ \rho S = a_{ij} e_{ij} + \gamma_{ij} \phi_{,j} + \gamma_{ij} \psi_{,j} + \frac{\rho C_{ir,\theta}}{T_{0,\theta}} + aC \] (6)

\[ \rho T_{ij} \dot{S} = -q_{i,j} \] (7)

\[ -\eta_{i,j} = \dot{\theta} \] (8)

\[ P = -b_{ij} e_{ij} - m \psi_{,j} + a \theta_{,j} + bC \] (9)

Equations of motion:

\[ \sigma_{ij,j} = \rho \ddot{x}_i \] (10)

Balance of equilibrated forces:

\[ \Omega_{i,j} + \xi = k_1 \ddot{\phi} \] (11)

\[ \chi_{i,j} + \zeta = k_2 \ddot{\psi} \] (12)

Equation of heat conduction:
\[ q_i + \tau \delta_i = -K_y \Theta_{ij} \]  \hspace{1cm} (13)

Equation of chemical potential:
\[ \eta_i + \tau \eta_i = -d_P \]  \hspace{1cm} (14)

In the equations mentioned above, \( \sigma_{ij}, e_{ij} \) are, respectively, the components of the stress and strain tensor, \( u_i \) are the components of displacement vector \( u_1, \Omega, \chi_i \) are equilibrated stress vectors, \( \xi, \zeta \) are the intrinsic equilibrated body forces associated to macro pores and fissures respectively, \( \rho \) is the density, \( C_e \) is the specific heat at the constant strain, \( q_1, \eta_i \) are the components of heat and mass diffusion flux vectors \( q, \eta \) respectively, \( k_1, k_2 \) are the coefficients of equilibrated inertia, \( T_0 \) is the absolute temperature in the reference state, \( \theta \) is the temperature variation from the absolute temperature \( T_0 \), \( C \) is the concentration of the diffusion material in the elastic body, \( S, P \) are entropy and chemical potential per unit mass respectively, \( \phi, \psi \) are change in volume fraction fields from the reference volume fraction, \( c_{ijkl} = c_{ijkl} \) is the tensor of elastic constants, \( K_y (=K_y \), \( d_y (=d_y \) are respectively the components of thermal conductivity and diffusivity, \( a_y, b_y \) are tensors of thermal and diffusion moduli respectively, \( a, b \) are, respectively, the coefficients describing the measure of thermodynamics and of mass diffusion effects, \( \tau_\theta \) is the thermal relaxation time which ensures that the heat conduction equation predicts finite speeds of heat propagation and \( \tau^0 \) is the diffusion relaxation time, which ensures that the equation satisfied by the concentration also predicts finite speeds of propagation of matter from one medium to the other.

If we put \( c_{ijkl} = \lambda \delta_i \delta_j + \mu \delta_i \delta_k + \mu \delta_i \delta_l + a_y = \zeta_i \delta_j, b_y = \zeta_j \delta_i, p_y = p_i \delta_j, q_y = q_j \delta_i, \alpha_y = \alpha_i \delta_j, f_y = f_j \delta_i, K_y = K \delta_i, d_y = D \delta_i \) in the above equations, then from Eqs. (10)-(14) with the aid of Eqs. (1)-(9), we obtain the basic equations for homogeneous isotropic thermoelastic diffusion material with double porosity as:

\[ \mu \Delta u + (\lambda + \mu) \text{grad div } u + p_1 \text{grad } \phi + p_2 \text{grad } \psi - \xi_i \text{grad } \xi_j \text{grad } C = \rho \dot{u} \]  \hspace{1cm} (15)

\[ -p_1 \text{div } u + (r_1 \Delta - d^*) \phi + (r_1 \Delta - \alpha_1) \psi + \gamma_1 \theta + vC = k_1 \dot{\phi} \]  \hspace{1cm} (16)

\[ -p_2 \text{div } u + (r_1 \Delta - \alpha_1) \phi + (r_2 \Delta - f) \psi + \gamma_2 \theta + mC = k_2 \dot{\psi} \]  \hspace{1cm} (17)

\[ \left( \frac{\partial}{\partial t} + \tau_\theta \frac{\partial^2}{\partial t^2} \right) \left( \rho C_e \theta + T_0 (\xi_i \text{div } u + \gamma_i \dot{\phi} + \gamma_i \dot{\psi} + aC) \right) = K \Delta \theta \]  \hspace{1cm} (18)

\[ D \Delta [-\xi_i \text{div } u - v \phi - m \psi - a \theta + bC] = \left( \frac{\partial}{\partial t} + \tau^0 \frac{\partial^2}{\partial t^2} \right) C \]  \hspace{1cm} (19)

In the upcoming sections, the chemical potential has been used as a state variable instead of the concentration. The Eqs. (15)-(19) can be rewritten as:

\[ \mu \Delta u + (\lambda + \mu) \text{grad div } u + g_1 \text{grad } \phi + g_2 \text{grad } \psi - s_i \text{grad } \theta - l_i \text{grad } P = \rho \dot{u} \]  \hspace{1cm} (20)

\[ -g_1 \text{div } u + (r_1 \Delta - d_1) \phi + (r_1 \Delta - e_1) \psi + \xi_1 \dot{\phi} + wP = k_1 \dot{\phi} \]  \hspace{1cm} (21)

\[ -g_2 \text{div } u + (r_1 \Delta - e_1) \phi + (r_2 \Delta - f_1) \psi + \xi_2 \dot{\psi} + vP = k_2 \dot{\psi} \]  \hspace{1cm} (22)
\[-\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)T \left[s_i \text{div} u + \xi_i \phi + \xi_{22} \psi + c^* \theta + sP\right] + K \Delta \theta = 0 \tag{23}\]

\[-\left(\frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2}\right)[l_i \text{div} u + w\phi + f'\psi + s\theta + nP] + D \Delta P = 0 \tag{24}\]

where

\[n = \frac{1}{b}, l_i = n\xi_i, s_i = \xi_i + al_i, g_i = p_i - vl_i, g_2 = p_2 - ml_i, \lambda' = \lambda - l_i\xi_2, s = an, \nu = mn, w = vn,\]

\[d_i = d' - wv, \epsilon_{11} = \alpha_i - wv, \xi_{11} = \gamma_i + vs, f_i = f - mv, \xi_{22} = \gamma_2 + ms, c^* = \frac{\rho C_p}{T_0} + as.\]

### 3 STEADY OSCILLATIONS

Now, we consider the case of steady oscillations. We assume the displacement vector, volume fraction fields, temperature change and chemical potential functions as:

\[\begin{align*}
[u(x,t), \phi(x,t), \psi(x,t), \theta(x,t), P(x,t)] &= \text{Re}[(u^*, \phi^*, \psi^*, \theta^*, P^*)e^{-ist}] 
\end{align*}\]

where, \(\omega\) is oscillation frequency.

Using Eq. (25) in Eqs. (20)–(24) and omitting asterisk (*) for simplicity, we obtain the system of equations of steady oscillations as:

\[\begin{align*}
\left[\mu \Delta + (\lambda' + \mu) \text{grad div} + \rho \omega^2 \right]u &+ g_i \text{grad} \phi + g_2 \text{grad} \psi - s_i \text{grad} \theta - l_i \text{grad} P = 0 \tag{26} \\
-g_i \text{div} u + [t_i \Delta - d_i + k_i \omega^2] \phi + (r_i \Delta - \epsilon_{i1}) \psi + \xi_{i1} \theta + wP &= 0 \tag{27} \\
-g_2 \text{div} u + (r_i \Delta - \epsilon_{i1}) \phi + [l_i \Delta - f_i + k_i \omega^2] \psi + \xi_{22} \theta + vP &= 0 \tag{28} \\
\tau_i T_0 [s_i \text{div} u + \xi_{11} \phi + \xi_{22} \psi] + [K \Delta + \tau_i T_0 c^*] \theta + \tau_i T_0 sP &= 0 \tag{29} \\
\tau^i [l_i \text{div} u + w\phi + f'\psi + s\theta] + [D \Delta + \tau^i n] P &= 0 \tag{30}
\end{align*}\]

where, \(\tau_i = \tau_0 (1 - \tau_0 \tau_0), \tau^i = \tau_0 (1 - \tau_0 \tau^0)\).

We introduce the second order matrix differential operators with constant coefficients

1) \(F(D_0) = \left\|F_{\rho_0}(D_0)\right\|_{\rho, \theta}

where

\[\begin{align*}
F_{\rho_0}(D_0) &= [\mu \Delta + \rho \omega^2] \rho + (\lambda' + \mu) \frac{\partial^2}{\partial \xi_1 \partial \xi_1}, F_{\rho_1}(D_0) = g_1 \frac{\partial}{\partial \xi_1}, F_{\rho_2}(D_0) = g_2 \frac{\partial}{\partial \xi_2}, F_{\rho_3}(D_0) = -s_i \frac{\partial}{\partial \xi_1}, F_{\rho_4}(D_0) = -l_i \frac{\partial}{\partial \xi_2}, \tag{26} \\
F_{\phi_1}(D_0) &= -g_i \frac{\partial}{\partial \xi_1}, F_{\phi_2}(D_0) = t_i \Delta - d_i + k_i \omega^2, F_{\phi_3}(D_0) = F_{\phi_4}(D_0) = F_{\phi_5}(D_0) = F_{\phi_6}(D_0) = F_{\phi_7}(D_0) = F_{\phi_8}(D_0) = w, F_{\phi_9}(D_0) = -g_2 \frac{\partial}{\partial \xi_2}. \tag{24}
\end{align*}\]
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\[ F_{34}(D_4) = t_1 \Delta - f_1 + k_1 \theta_1, F_{35}(D_5) = \xi_{23}, F_{33}(D_3) = v, F_{34}(D_4) = \tau_s T_1, \frac{\partial}{\partial x_1}, F_{32}(D_2) = \tau_{11} T_1, F_{45}(D_5) = \tau_s T_2, \frac{\partial}{\partial x_2}, F_{44}(D_4) = \tau_{11} T_2, \frac{\partial}{\partial x_3}, F_{43}(D_3) = \tau_s T_3, \frac{\partial}{\partial x_4}, F_{42}(D_2) = \tau_{11} T_3, \frac{\partial}{\partial x_5}, F_{55}(D_5) = \tau_s T_4, \frac{\partial}{\partial x_6}, F_{54}(D_4) = \tau_{11} T_4, \frac{\partial}{\partial x_7}, F_{53}(D_3) = \tau_s T_5, \frac{\partial}{\partial x_8}, F_{52}(D_2) = \tau_{11} T_5, \frac{\partial}{\partial x_9}. \]

\[ F_{45}(D_5) = K \Delta + \tau_s T_1, F_{44}(D_4) = \tau_s T_2, F_{43}(D_3) = \tau_s T_3, F_{55}(D_5) = \tau_s T_4, F_{54}(D_4) = \tau_s T_5, F_{53}(D_3) = \tau_s T_6, F_{52}(D_2) = \tau_s T_7. \]

2) \[ F'(D_x) = \left\| F_{gh}(D_x) \right\|_{p,q} \]

where

\[ F_{pq}(D_x) = \mu \Delta, F_{qq}(D_x) = 0, F_{pp}(D_x) = 0, F_{pq}(D_x) = 0, F_{pp}(D_x) = \rho, F_{pq}(D_x) = F_{pp}(D_x) = 0. \]

\[ F_{35}(D_5) = t_2 \Delta, F_{34}(D_4) = K \Delta, F_{33}(D_3) = 0, F_{55}(D_5) = 0, F_{45}(D_5) = 0, F_{45}(D_5) = D \Delta, p, q = 1, 2, 3. \]

The system of Eqs. (26)-(30) can be written as:

\[ F(D_x) U(x) = 0 \]

where \( U = (u, \phi, \psi, \theta, P) \) is a seven-component vector function for \( E^3 \). The matrix \( F'(D_x) \) is called the principal part of the operator \( F(D_x) \).

Definition 1: The operator \( F(D_x) \) is said to be elliptic if \( \text{det} F'(\xi) \neq 0 \), where \( \xi = (\kappa_1, \kappa_2, \kappa_3) \).

We have, \( \text{det} F'(\xi) = \det \begin{pmatrix}
\mu |\kappa|^2 + (\lambda' + \mu) \kappa_1 \kappa_2 & (\lambda' + \mu) \kappa_1 \kappa_3 & 0 & 0 & 0 & 0 \\
(\lambda' + \mu) \kappa_1 \kappa_2 & \mu |\kappa|^2 + (\lambda' + \mu) \kappa_2 \kappa_3 & 0 & 0 & 0 & 0 \\
(\lambda' + \mu) \kappa_1 \kappa_3 & (\lambda' + \mu) \kappa_2 \kappa_3 & \mu |\kappa|^2 + (\lambda' + \mu) \kappa_3^2 & 0 & 0 & 0 \\
0 & 0 & 0 & t_1 |\kappa|^2 & \tau_1 |\kappa|^2 & 0 & 0 \\
0 & 0 & 0 & \tau_1 |\kappa|^2 & t_2 |\kappa|^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K |\kappa|^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & D |\kappa|^2 \\
\end{pmatrix} \]

\[ = \mu^2 (\lambda' + 2 \mu) KD (t_1 t_2 - r_2^2) |\kappa|^4 \]

Therefore, operator \( F(D_x) \) is an elliptic differential operator if and only if

\[ \mu (\lambda' + 2 \mu) KD (t_1 t_2 - r_2^2) \neq 0 \] (31)

Definition 2: The fundamental solution of the system of Eqs. (26)-(30) (the fundamental matrix of operator \( F(D_x) \)) is the matrix \( G(x) = \left\| G_{gh}(x) \right\|_{p,q} \) satisfying condition

\[ F(D_x) G(x) = \delta(x) I(x) \] (32)

where \( \delta(x) \) is the Dirac delta, \( I(x) = \left\| I_{gh} \right\|_{p,q} \) is the unit matrix and \( x \in E^3 \). Now, we construct \( G(x) \) in terms of elementary functions.
4 Fundamental Solution of a System of Equations of Steady Oscillations

Let us consider the system of non-homogeneous equations

\[ [\mu \Delta + (\lambda' + \mu) \text{grad} + \rho \omega^2] \mathbf{u} - g_1 \text{grad} \phi - g_2 \text{grad} \psi + \tau_i T_{ij} s_{ij} \text{grad} \theta + \tau^i l_{ij} \text{grad} \mathbf{P} = \mathbf{H} \]  \hspace{1cm} (33)

\[ g_i \text{div} \mathbf{u} + [t_i \Delta - d_i + k_i \omega^2] \phi + [t_i \Delta - e_{i1}] \psi + \tau_i T_{ij} s_{ij} \theta + \tau^i w = L \]  \hspace{1cm} (34)

\[ g_2 \text{div} \mathbf{u} + [t_i \Delta - f_i + k_i \omega^2] \psi + \tau_i T_{ij} s_{ij} \theta + \tau^i v = M \]  \hspace{1cm} (35)

\[ -s_i \text{div} \mathbf{u} + \xi_{i1} \phi + \xi_{i2} \psi + [K \Delta + \tau_i T_{0i} c^*] \theta + \tau^i s = Z \]  \hspace{1cm} (36)

\[ -l_i \text{div} \mathbf{u} + w \phi + v \psi + \tau_i T_{0i} s \theta + [D \Delta + \tau^i n] = X \]  \hspace{1cm} (37)

where \( \mathbf{H} \) is three-component vector function on \( E^3 \), \( L, M, Z \) and \( X \) are scalar functions on \( E^3 \).

The system of Eqs. (33)–(37) may be written in the form

\[ \mathbf{F}^\tau (\mathbf{D}) \mathbf{u}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) \]  \hspace{1cm} (38)

where \( \mathbf{F}^\tau \) is the transpose of matrix \( \mathbf{F} \), \( \mathbf{Q} = (\mathbf{H}, L, M, Z, X) \), \( \mathbf{x} \in E^3 \).

Applying the operator div to Eq. (33), we obtain

\[ [(\lambda' + 2\mu) \Delta + \rho \omega^2] \text{div} \mathbf{u} - g_1 \Delta \phi - g_2 \Delta \psi + \tau_i T_{ij} s_{ij} \Delta \theta + \tau^i l_{ij} \Delta \mathbf{P} = \text{div} \mathbf{H} \]  \hspace{1cm} (39)

The Eqs. (34)–(37) and (39) may be written in the form

\[ \mathbf{N}(\Delta) \mathbf{S} = \mathbf{Q} \]  \hspace{1cm} (40)

where \( \mathbf{S} = (\text{div} \mathbf{u}, \phi, \psi, \theta, \mathbf{P}) \), \( \mathbf{Q} = (w_1, w_2, w_3, w_4, w_5) = (\text{div} \mathbf{H}, L, M, Z, X) \) and

\[ \mathbf{N}(\Delta) = \left| \begin{array}{cccccc}
\lambda' + 2\mu & \rho \omega^2 & -g_1 & -g_2 & \tau_i T_{ij} s_{ij} & \tau^i l_{ij} \\
g_1 & t_i \Delta - d_i & k_i \omega^2 & r_i \Delta - e_{i1} & \tau_i T_{ij} s_{ij} & \tau^i w \\
g_2 & r_i \Delta - e_{i1} & t_i \Delta - f_i & k_i \omega^2 & \tau_i T_{ij} s_{ij} & \tau^i v \\
-s_i & \xi_{i1} & \xi_{i2} & K \Delta + \tau_i T_{0i} c^* & \tau^i s \\
-l_i & w & v & \tau_i T_{0i} s & D \Delta + \tau^i n
\end{array} \right| \]  \hspace{1cm} (41)

Eqs. (34)–(37) and (39) may be also written as:

\[ \Gamma(\Delta) \mathbf{S} = \mathbf{\Psi} \]  \hspace{1cm} (42)

where

\[ \mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5), \Psi_p = \frac{1}{M} \sum_{i=1}^{5} N_{\psi_i}^p w_i, \Gamma(\Delta) = \frac{\det \mathbf{N}(\Delta)}{M'}, \]  \hspace{1cm} (43)

\[ M' = (\lambda' + 2\mu) K \mathcal{D} (t_i t_j - r_i^2) \]  \hspace{1cm} p = 1, ..., 5.

and \( N_{\psi_i}^p \) is the cofactor of the element \( N_{\psi_i} \) of the matrix \( \mathbf{N} \). From Eqs. (41) and (43), we see that
\[ \Gamma(\Delta) = \sum_{i=1}^{5} \Gamma(\Delta + \lambda_i^2) \]

where \( \lambda_i^2, i = 1, \ldots, 5 \) are the roots of the equation \( \Gamma(-\kappa) = 0 \), w.r.t. \( \kappa \).

Applying operator \( \Gamma(\Delta) \) to Eq. (33), we get

\[ \Gamma(\Delta)(\Delta + \lambda_i^2)\mathbf{u} = \Psi' \]  

(44)

where \( \lambda_i^2 = \frac{\rho \omega^2}{\mu} \) and

\[ \Psi' = \frac{1}{\mu} \left[ \Gamma(\Delta) \mathbf{H} - \text{grad} \left[ (\lambda'_i + \mu)\Psi_i - g_i \Psi_i z - g_j \Psi_j z + r_i T_0 s_i \Psi_i + r'_i \psi_i \Psi_i \right] \right] \]  

(45)

From Eqs. (42) and (44), we have

\[ \Theta(\Delta)\mathbf{U}(\mathbf{x}) = \Psi(\mathbf{x}) \]  

(46)

where

\[ \dot{\Psi}(\mathbf{x}) = (\Psi'_1, \Psi'_2, \Psi'_3, \Psi'_4, \Psi'_5) \quad \text{and} \quad \Theta(\Delta) = \left[ \Theta_{pq}(\Delta) \right]_{p,q=0,1,2,3,4,5,6} \]

\[ \Theta_{ii}(\Delta) = \Gamma(\Delta)(\Delta + \lambda_i^2), \Theta_{ij}(\Delta) = \Gamma(\Delta), \Theta_{pq}(\Delta) = 0 \quad i = 1, 2, 3 \quad j = 4, \ldots, 7 \quad p, q = 1, \ldots, 7 \quad p \neq q \]

Eqs. (43) and (45) can be rewritten in the form

\[ \Psi' = \frac{1}{\mu} \left[ \Gamma(\Delta) \mathbf{J} + w_{i1}(\Delta) \text{grad div} \mathbf{H} + w_{i2}(\Delta) \text{grad grad div} \mathbf{H} + w_{i3}(\Delta) \text{grad M} + w_{i4}(\Delta) \text{grad Z} + w_{i5}(\Delta) \text{grad X}, \right. \]

\[ \Psi'_2 = w_{i2}(\Delta) \text{div} \mathbf{H} + w_{i2}(\Delta) L + w_{i2}(\Delta) M + w_{i4}(\Delta) Z + w_{i5}(\Delta) X, \]

\[ \Psi'_3 = w_{i3}(\Delta) \text{div} \mathbf{H} + w_{i3}(\Delta) L + w_{i3}(\Delta) M + w_{i4}(\Delta) Z + w_{i5}(\Delta) X, \]

\[ \Psi'_4 = w_{i4}(\Delta) \text{div} \mathbf{H} + w_{i4}(\Delta) L + w_{i4}(\Delta) M + w_{i4}(\Delta) Z + w_{i5}(\Delta) X, \]

\[ \Psi'_5 = w_{i5}(\Delta) \text{div} \mathbf{H} + w_{i5}(\Delta) L + w_{i5}(\Delta) M + w_{i5}(\Delta) Z + w_{i5}(\Delta) X, \]

(47)

where \( \mathbf{J} = \left[ \delta_{pq} \right]_{p,q=1,2,3} \) is the unit matrix.

In the above equation, the following notations have been used:

\[ w_{i1}(\Delta) = -\frac{1}{\mu M^2} \left[ (\lambda'_i + \mu)N_{i1}^r(\Delta) - g_i N_{i2}^r(\Delta) - g_j N_{i3}^r(\Delta) + r_i T_0 s_i N_{i4}^r(\Delta) + r'_i N_{i5}^r(\Delta) \right], \]

\[ w_{i2}(\Delta) = \frac{N_{i2}^r(\Delta)}{M^2} \quad p = 1, \ldots, 5 \quad j = 2, \ldots, 5. \]

From Eq. (47), we have

\[ \dot{\Psi}(\mathbf{x}) = \mathbf{R}''(\mathbf{D}_x) \mathbf{Q}(\mathbf{x}) \]  

(48)

where \( \mathbf{R}(\mathbf{D}_x) = \left[ R_{pg}(\mathbf{D}_x) \right]_{p,q=0,1,2,3,4,5,6} \).
\[ R_i(\mathbf{D}_x) = \frac{1}{\mu} \Gamma(\Delta) \delta_{ij} + w_{ij}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, R_{p+1}(\mathbf{D}_x) = w_{ij}(\Delta) \frac{\partial}{\partial x_i}, R_{p+2}(\mathbf{D}_x) = w_{ij}(\Delta) \frac{\partial}{\partial x_i} \]

\[ R_{p+2}(\mathbf{D}_x) = w_{ij}(\Delta) \quad i, j = 1, 2, 3, \quad p, l = 2, 3, 4, 5. \quad (49) \]

From Eqs. (38), (46) and (48), we obtain \( \Theta U = R' U = R' F^\mu U \). It implies that

\[ R' F^\mu = \Theta, \quad F(\mathbf{D}_x)R(\mathbf{D}_x) = \Theta(\Delta) \quad (50) \]

We assume that \( \lambda_{ij} \neq \lambda_{ij}^2 \neq 0 \), \( p, q = 1, ..., 6 \) and \( M' \mu \neq 0 \). Let

\[
Y(x) = \left| Y_p(x) \right|_{p=1}^{p=6}, \quad Y_{pp}(x) = \sum_{g=1}^{6} r_{gi} \delta_{ij}(x), \quad Y_{pp}(x) = \sum_{g=1}^{6} r_{gi} \delta_{ij}(x), \quad Y_{pp}(x) = \sum_{g=1}^{6} r_{gi} \delta_{ij}(x).
\]

\[ Y_{pp}(x) = 0 \quad p = 1, 2, 3, e = 4, ..., 7, \quad q, h = 1, ..., 7, \quad q \neq h \quad (51) \]

where

\[
\delta_{ij}(x) = -\frac{e^{2i|\mathbf{x}|}}{4\pi |\mathbf{x}|} \delta_{ij} = \prod_{i=1}^{6} (\lambda_i^2 - \lambda_i^2)^{-1}, \quad \delta_{ij} = \prod_{i=1}^{6} (\lambda_i^2 - \lambda_i^2)^{-1} \quad g = 1, ..., 6, \quad h = 1, ..., 5. \quad (52) \]

Lemma 1: The matrix \( Y \) is the fundamental matrix of the operator \( \Theta(\Delta) \) i.e.

\[ \Theta(\Delta)Y(x) = \delta(x) \quad (53) \]

To prove the lemma, it is sufficient to prove that

\[ \Gamma(\Delta + \lambda_i^2)Y_{pp}(x) = \delta(x), \quad \Gamma(\Delta)Y_{pp}(x) = \delta(x) \quad (54) \]

Consider \( \sum_{i=1}^{6} r_{ii} = \frac{1}{2} \sum_{j=1}^{6} (-1)^j z_j \), where

\[
z_1 = \prod_{i=1}^{6} (\lambda_i^2 - \lambda_i^2) \prod_{j=1}^{6} (\lambda_j^2 - \lambda_j^2) \prod_{p=1}^{6} (\lambda_p^2 - \lambda_p^2), \quad z_2 = \prod_{i=1}^{6} (\lambda_i^2 - \lambda_i^2) \prod_{j=1}^{6} (\lambda_j^2 - \lambda_j^2) \prod_{p=1}^{6} (\lambda_p^2 - \lambda_p^2), \quad z_3 = \prod_{i=1}^{6} (\lambda_i^2 - \lambda_i^2) \prod_{j=1}^{6} (\lambda_j^2 - \lambda_j^2) \prod_{p=1}^{6} (\lambda_p^2 - \lambda_p^2), \quad z_4 = \prod_{i=1}^{6} (\lambda_i^2 - \lambda_i^2) \prod_{j=1}^{6} (\lambda_j^2 - \lambda_j^2) \prod_{p=1}^{6} (\lambda_p^2 - \lambda_p^2), \quad z_5 = \prod_{i=1}^{6} (\lambda_i^2 - \lambda_i^2) \prod_{j=1}^{6} (\lambda_j^2 - \lambda_j^2) \prod_{p=1}^{6} (\lambda_p^2 - \lambda_p^2), \quad z_6 = \prod_{i=1}^{6} (\lambda_i^2 - \lambda_i^2) \prod_{j=1}^{6} (\lambda_j^2 - \lambda_j^2) \prod_{p=1}^{6} (\lambda_p^2 - \lambda_p^2).
\]

Upon simplifying the R.H.S. of above relation, we obtain

\[ \sum_{i=1}^{6} r_{ii} = 0. \quad (55) \]

Similarly, we find that
\[
\sum_{i=2}^{6} r_{1i}(\lambda_i^2 - \lambda_1^2) = 0, \quad \sum_{i=2}^{6} r_{1i} \left[ \Pi_j (\lambda_i^2 - \lambda_1^2) \right] = 0, \quad \sum_{i=2}^{6} r_{1i} \left[ \Pi_i (\lambda_j^2 - \lambda_1^2) \right] = 0, \quad \sum_{i=2}^{6} r_{1i} \left[ \Pi_j (\lambda_i^2 - \lambda_1^2) \right] = 0. \quad (56)
\]

Also, we have
\[
r_{1i} \left[ \Pi_j (\lambda_i^2 - \lambda_1^2) \right] = 1, (\lambda + \lambda_1^2) \zeta_{ge}(x) = \delta(x) + (\lambda_1^2 - \lambda_g^2) \zeta_{ge}(x) \quad \mu, g = 1, \ldots, 6
\]

Now, let us consider
\[
\Gamma(\Delta)(\Delta + \lambda_1^2) Y_{\gamma i}(x) = \Pi(\Delta + \lambda_1^2) \sum_{g=1}^{6} r_{1g} \zeta_{ge}(x) = \Pi(\Delta + \lambda_1^2) \sum_{g=1}^{6} r_{1g} [\delta(x) + (\lambda_1^2 - \lambda_g^2) \zeta_{ge}(x)]
\]

Using Eqs. (55)-(57) in the above relation (58), we obtain
\[
\Gamma(\Delta)(\Delta + \lambda_1^2) Y_{\gamma i}(x) = \Pi(\Delta + \lambda_1^2) \sum_{g=1}^{6} r_{1g} \zeta_{ge}(x) = \Pi(\Delta + \lambda_1^2) \sum_{g=1}^{6} r_{1g} \zeta_{ge}(x) = \Pi(\Delta + \lambda_1^2) \sum_{g=1}^{6} r_{1g} \zeta_{ge}(x) = \Pi(\Delta + \lambda_1^2) \zeta_{ge}(x) = \delta(x).
\]

Similarly to Eqs. (55)-(57), we obtain
\[
\sum_{i=2}^{5} r_{2i} = 0, \quad \sum_{i=2}^{5} r_{2i} (\lambda_i^2 - \lambda_2^2) = 0, \quad \sum_{i=2}^{5} r_{2i} \left[ \Pi_j (\lambda_i^2 - \lambda_2^2) \right] = 0, \quad \sum_{i=2}^{5} r_{2i} \left[ \Pi_i (\lambda_j^2 - \lambda_2^2) \right] = 0, \quad \sum_{i=2}^{5} r_{2i} \left[ \Pi_j (\lambda_i^2 - \lambda_2^2) \right] = 0. \quad (59)
\]

Now, we consider the Eq. (54),
\[
\Gamma(\Delta) Y_{\mu i}(x) = \Pi(\Delta + \lambda_1^2) \sum_{g=1}^{6} r_{2g} \zeta_{ge}(x) = \Pi(\Delta + \lambda_1^2) \sum_{g=1}^{6} r_{2g} [\delta(x) + (\lambda_1^2 - \lambda_g^2) \zeta_{ge}(x)]
\]

We introduce the matrix
\[
G(x) = R(D_x) Y(x)
\]

From Eqs. (50), (53) and (60), we obtain
\[
F(D_x) G(x) = F(D_x) R(D_x) Y(x) = \Theta(\Delta) Y(x) = \delta(x) I(x)
\]
Therefore, \( \mathbf{G}(\mathbf{x}) \) is a solution to Eq. (32).

Theorem 1: If the condition (31) is satisfied, then the matrix \( \mathbf{G}(\mathbf{x}) \) defined by the Eq. (60) is the fundamental solution of the system Eqs. (26)-(30) and each element \( G_{gh}(\mathbf{x}) \) of the matrix \( \mathbf{G}(\mathbf{x}) \) is represented in the following form:

\[
G_{gh}(\mathbf{x}) = R_{gh}(\mathbf{D}_x)Y_{1i}(\mathbf{x}), \quad G_{gh}(\mathbf{x}) = R_{gh}(\mathbf{D}_x)Y_{4i}(\mathbf{x}) \quad g = 1,\ldots,7 \quad h = 1,2,3 \quad q = 4,\ldots,7
\]

(61)

5 BASIC PROPERTIES OF FUNDAMENTAL SOLUTIONS

Theorem 2: Each column of the matrix \( \mathbf{G}(\mathbf{x}) \) is a solution of system of Eqs. (26)-(30) at every point \( \mathbf{x} \in \mathbb{R}^3 \) except at the origin.

Theorem 3: If the condition (31) is satisfied, then the fundamental solution of the system \( \mathbf{F}'(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{0} \), is the matrix \( \mathbf{G}'(\mathbf{x}) = [G'_{gh}(\mathbf{x})]_{r,s} \), where

\[
G'_{gh}(\mathbf{x}) = \frac{1}{\mu} \left[ \Delta \delta_{gh} - \frac{\lambda' + \mu}{\lambda' + 2\mu} \mathbf{\partial}^2 \right]_{\mathbf{x}} Y_{1i}^*(\mathbf{x}) = \frac{1}{\mu} R_{gh}^* Y_{1i}^*(\mathbf{x}) = \lambda' \cdot \delta_{gh} + \mu \cdot \frac{x_g x_h}{|\mathbf{x}|},
\]

\[
G'_{gh}(\mathbf{x}) = \frac{1}{K} Y_{4i}^*(\mathbf{x}), \quad G'_{gh}(\mathbf{x}) = \frac{1}{D} Y_{4i}^*(\mathbf{x}), \quad G_{gh+1,k}(\mathbf{x}) = G_{gh+2,k}(\mathbf{x}) = G_{gh}(\mathbf{x}) = 0,
\]

\[
G'_{gh}(\mathbf{x}) = 0, \quad Y_{1i}^*(\mathbf{x}) = -\frac{\mathbf{x}_g}{8\pi|\mathbf{x}|^3}, \quad Y_{4i}^*(\mathbf{x}) = -\frac{1}{8\pi|\mathbf{x}|^3} \lambda' = -\frac{\lambda' + 3\mu}{8\pi\mu(\lambda' + 2\mu)}, \quad \lambda' = -\frac{\lambda' + \mu}{8\pi\mu(\lambda' + 2\mu)}
\]

(62)

\[
R'_{gh} = \frac{\mathbf{\partial}^2}{\mathbf{x}_g \mathbf{x}_h} - \Delta \delta_{gh}, \quad g, h = 1, 2, 3 \quad p = 1,\ldots, 6 \quad q = 4, 5.
\]

Corollary 1: The relations

\[
G'_{gh}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \quad G'_{gh}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right)
\]

hold in the neighbourhood of the origin, where \( g, h = 1, 2, 3 \) and \( p, q = 4, 5, 6, 7 \).

Lemma 2: If the condition (31) is satisfied, then

\[
\Delta w_{pl}(\Delta) = -\frac{1}{\mu} \Gamma(\Delta) \delta_{pl} + \frac{1}{M'} (\Delta + \lambda' \cdot \mu) N_{p1}(\Delta) \quad p = 1,\ldots, 5
\]

(63)

We will prove the result for \( p = 1 \). For \( p = 1 \),

\[
w_{1i}(\Delta) = -\frac{1}{M' \mu} \left[ (\lambda' + \mu) N_{11}(\Delta) - g_1 N_{12}(\Delta) - g_2 N_{13}(\Delta) + \tau_1 T_0 s_1 N_{14}(\Delta) + r_l' s_3 S_{15}(\Delta) \right]
\]

Now

\[
\Gamma(\Delta) = \frac{1}{M'} \text{det} \mathbf{N}(\Delta) = \frac{1}{M'} \left[ (\Delta + \lambda' \cdot \mu) N_{11}(\Delta) - g_1 N_{12}(\Delta) - g_2 N_{13}(\Delta) \right. \]

\[
+ \tau_1 T_0 s_1 N_{14}(\Delta) + r_l' s_3 S_{15}(\Delta)
\]

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Therefore,
\[ \Delta w_1(\Delta) = -\frac{1}{M'\mu} \left[ (\lambda' + \mu)\Delta N'_{1}(\Delta) - g_i\Delta N'_i(\Delta) - g_{ij}\Delta N'_{ij}(\Delta) + \tau_{ij}s_{ij}\Delta N'_{ij}(\Delta) + \tau'_{ij}\Delta N'_{ij}(\Delta) \right] \]
\[ = -\frac{1}{M'\mu} \left[ \Gamma(\Delta) M' - (\mu \Delta + \rho \omega^2) N'_1(\Delta) \right] = -\frac{\Gamma(\Delta) + (\Delta + \lambda'_p)}{M'\mu} N'_1(\Delta) \]

The results for \( p = 2, \ldots, 5 \) can be proved in the similar manner. Let
\[ c_{p11} = -\frac{1}{M'\lambda'_p} r_{p2} N'_1(-\lambda'_p), c_{611} = \frac{1}{\rho \omega^2} c_{p11} = r_p w_{2g}(-\lambda'_p), c_{p1g} = r_{p2} w_{1g}(-\lambda'_p), \]
\[ c_{pgh} = r_{p2} w_{gh}(-\lambda'_p) \quad p = 1, \ldots, 5 \quad g, h = 2, \ldots, 5. \] (64)

Theorem 4: If \( \mathbf{x} \neq \mathbf{0} \), then
\[ G_{gh}(\mathbf{x}) = \sum_{p=1}^{5} \sum_{\tau} c_{p1} \xi_{p,gh}(\mathbf{x}) + c_{611} R'_{gh} \zeta_{b}(\mathbf{x}), G_{e_{x}z_{y}}(\mathbf{x}) = \sum_{p=1}^{5} c_{p1e} \xi_{p,e}(\mathbf{x}), \]
\[ G_{e_{xy}z_{y}}(\mathbf{x}) = \sum_{p=1}^{5} c_{p1y} \xi_{p,y}(\mathbf{x}), G_{e_{xy}z_{x}}(\mathbf{x}) = \sum_{p=1}^{5} c_{p1y} \xi_{p,y}(\mathbf{x}) \quad g, h = 1, 2, 3 \quad e, y = 2, \ldots, 5. \] (65)

From Eq. (57),
\[ \Delta \tau_{p}(\mathbf{x}) = -\lambda'_p \zeta_{p}(\mathbf{x}) \] (66)

Thus, we have
\[ \delta_{gh} \tau_{p}(\mathbf{x}) = -\frac{1}{\lambda'_p} \left[ \frac{1}{\lambda'_p} \Delta \delta_{gh} \tau_{p} (\mathbf{x}) \right] = -\frac{1}{\lambda'_p} \left[ \frac{\partial^2}{\partial x_g \partial x_h} - R'_{gh} \right] \zeta_{p}(\mathbf{x}) \] (67)

From Eq. (63), we get
\[ w_{1i}(-\lambda'_p) - \frac{\Gamma(-\lambda'_p)}{\mu \lambda'_p} = -\frac{1}{M' \lambda'_p} (\lambda'_p^2 - \lambda'_p^2) N'_i(-\lambda'_p) \quad p = 1, \ldots, 5 \] (68)

From Eq. (61) with the aid of Eqs. (49), (52) and (66)-(68), we have
\[ G_{gh}(\mathbf{x}) = R_{gh}(\mathbf{D}) Y_{11}(\mathbf{x}) = \left[ \frac{1}{\mu} \Gamma(\Delta) \delta_{gh} + w_{1i}(\Delta) \frac{\partial^2}{\partial x_g \partial x_h} \right] \sum_{p=1}^{6} r_{p1} \zeta_{p}(\mathbf{x}) \]
\[ = \sum_{p=1}^{6} r_{p1} \left[ \frac{1}{\mu} \Gamma(-\lambda'_p) \delta_{gh} + w_{1i}(\lambda'_p) \frac{\partial^2}{\partial x_g \partial x_h} \right] \zeta_{p}(\mathbf{x}) \] (69)
\[ = \sum_{p=1}^{6} r_{p1} \left[ -\frac{1}{\mu \lambda'_p} \Gamma(-\lambda'_p) \left( \frac{\partial^2}{\partial x_g \partial x_h} - R'_{gh} \right) + w_{1i}(\lambda'_p) \frac{\partial^2}{\partial x_g \partial x_h} \right] \zeta_{p}(\mathbf{x}) \]
\[ = \sum_{p=1}^{6} r_{p1} \left[ -\frac{1}{\mu \lambda'_p} (\lambda'_p^2 - \lambda'_p^2) N'_i(-\lambda'_p) \frac{\partial^2}{\partial x_g \partial x_h} + \frac{1}{\mu} \Gamma(-\lambda'_p) R'_{gh} \right] \zeta_{p}(\mathbf{x}) \]

Now,
\( (\lambda_6^2 - \lambda_h^2) r_{th} = r_{zh}, \ h = 1, \ldots, 5 \quad \text{and} \quad r_{ip} \Gamma (\lambda_p^2) = \begin{cases} 0, & \text{for} \ p = 1, \ldots, 5 \\ 1, & \text{for} \ p = 6 \end{cases} \) \hfill (70)

By virtue of Eqs. (64) and (70), Eq. (69) becomes

\[
G_{gh} (x) = -\sum_{p=1}^{5} r_{ip} N_{11}^* (\lambda_p^2) \frac{\partial^2}{\partial x_p \partial x_{-p}} \varphi_p (x) + \sum_{p=1}^{5} \mu \frac{1}{\mu} r_{ip} \Gamma (\lambda_p^2) R_{gh} (x) = \sum_{p=1}^{5} c_{p11} \varphi_{p, gh} (x) + c_{61} R_{gh} \varphi_6 (x)
\]

Now, consider

\[
G_{e+\nu+2} (x) = R_{e+\nu+2} (D_1) Y_{\nu} (x) = w_{\nu} (\Lambda) \sum_{p=1}^{4} r_{p} \varphi_p (x) + \sum_{p=1}^{5} w_{\nu} (\Lambda) \varphi_{p, \nu} (x) = \sum_{p=1}^{4} c_{p\nu} \varphi_p (x)
\]

Similarly the formula for \( G_{e+2, \nu} (x) \) can be proved.

Lemma 3: If the condition (31) is satisfied, then

\[
\sum_{p=1}^{5} r_{p} = \sum_{p=1}^{5} \lambda_p^2 = \sum_{p=1}^{5} \lambda_p^4 = 0, \sum_{p=1}^{5} \lambda_p^6 = 1, \sum_{p=1}^{5} \lambda_p^8 = 1, \sum_{p=1}^{5} \lambda_p^8 = 1, \sum_{p=1}^{5} \lambda_p^8 = 1, \sum_{p=1}^{5} \lambda_p^8 = 1, \sum_{p=1}^{5} \lambda_p^8 = 1, \sum_{p=1}^{5} \lambda_p^8 = \frac{M^*}{\rho \lambda^2 N_{11}^* (0)} \] \hfill (71)

and

\[
\sum_{p=1}^{5} c_{p11} = -\frac{1}{\rho \lambda^2}, \sum_{p=1}^{5} \lambda_p^2 c_{p11} = -\frac{1}{\lambda^4 + 2 \mu} \] \hfill (72)

Using Eq. (52), relations (71) can be proved by direct calculations.

\[
N_{11}^* (\lambda_p^2) = KD (t, t_2 - r_1^2) \lambda_p^8 + M_1 \lambda_p^6 + M_2 \lambda_p^4 + M_3 \lambda_p^2 + N_{11}^* (0), \ p = 1, \ldots, 5. \] \hfill (73)

where \( M_1, M_2, \) and \( M_3 \) are coefficients, independent of \( \lambda_p \) and skipped due to lengthy calculations.

From Eqs. (71) and (73), we obtain

\[
\sum_{p=1}^{5} \frac{1}{\lambda_p^2} r_{p} N_{11}^* (\lambda_p^2) = \sum_{p=1}^{5} r_{p} \left[ KD (t, t_2 - r_1^2) \lambda_p^8 + M_1 \lambda_p^6 + M_2 \lambda_p^4 + M_3 \lambda_p^2 + N_{11}^* (0) \right] = N_{11}^* (0) \sum_{p=1}^{5} \frac{1}{\lambda_p^2} r_{p} = \frac{M^*}{\rho \lambda^2}
\]

and

\[
\sum_{p=1}^{5} r_{p} N_{11}^* (\lambda_p^2) = \sum_{p=1}^{5} r_{p} \left[ KD (t, t_2 - r_1^2) \lambda_p^8 + M_1 \lambda_p^6 + M_2 \lambda_p^4 + M_3 \lambda_p^2 + N_{11}^* (0) \right] = KD (t, t_2 - r_1^2)
\]

Therefore, from (64), we have

\[
\sum_{p=1}^{5} c_{p11} = -\frac{1}{M^*} \sum_{p=1}^{5} \frac{1}{\lambda_p^2} r_{p} N_{11}^* (\lambda_p^2) = -\frac{1}{M^*} \sum_{p=1}^{5} \lambda_p^2 c_{p11} = -\frac{1}{M^*} \sum_{p=1}^{5} r_{p} N_{11}^* (\lambda_p^2) = -KD (t, t_2 - r_1^2) = -\frac{1}{(\lambda^4 + 2 \mu)}
\]
Theorem 5: The relations

\[ G_y(x) - G'_y(x) = \text{constant} + O(|x|) \]  

(74)

hold in neighborhood of the origin, where \( i, j = 1, \ldots, 7 \). Let \( x \neq 0 \). From Eqs. (62) and (65), we have

\[ G_y(x) - G'_y(x) = \frac{\partial^2}{\partial x_i \partial x_j} \sum_{p=0}^8 c_{p11} \varphi_p(x) + c_{p11} R' y_p(x) - \frac{1}{\mu} \left[ \varDelta \delta_{ij} - \left( 1 - \frac{\mu}{\lambda' + 2 \mu} \right) \frac{\partial^2}{\partial x_i \partial x_j} \right] Y''_{11}(x) \]

\[ = \frac{\partial^2}{\partial x_i \partial x_j} \left[ \sum_{p=0}^8 c_{p11} \varphi_p(x) - \frac{1}{\lambda' + 2 \mu} Y''_{11}(x) \right] + R'_{ij} \left[ \frac{1}{\rho \omega^2} \varphi_0(x) + \frac{1}{\mu} Y''_{11}(x) \right] \]  

(75)

For \( i, j = 1, 2, 3 \). In the neighborhood of the origin, from Eq. (52), we have

\[ \varphi_p(x) = -\frac{1}{4\pi|x|} \sum_{k=0}^p \left( \frac{i \lambda \varphi_k}{4\pi} \right)^p = Y''_{11}(x) - \frac{\lambda^2}{4\pi} Y''_{11}(x) + \varphi_{pp}(x) \]  

(76)

where \( \varphi_{pp}(x) = -\frac{1}{4\pi|x|} \sum_{k=1}^p \left( \frac{i \lambda \varphi_k}{4\pi} \right)^p \). Clearly,

\[ \varphi_{pp}(x) = O(|x|^2), \varphi_{pp}(x) = O(|x|), \varphi_{pp}(x) = \text{constant} + O(|x|) \]  

\( i, j = 1, 2, 3 \) \( p = 1, \ldots, 6 \). Clearly,

\[ \varphi_{pp}(x) = O(|x|^2), \varphi_{pp}(x) = O(|x|), \varphi_{pp}(x) = \text{constant} + O(|x|) \]  

\( i, j = 1, 2, 3 \) \( p = 1, \ldots, 6 \)

Consider

\[ \sum_{p=0}^8 c_{p11} \varphi_p(x) - \frac{1}{\lambda' + 2 \mu} Y''_{11}(x) = \sum_{p=0}^8 c_{p11} \left[ Y''_{11}(x) - \frac{i \lambda}{4\pi} + \varphi_{pp}(x) \right] - \sum_{p=1}^8 \lambda^2 c_{p11} + \frac{1}{\lambda' + 2 \mu} Y''_{11}(x) \]  

(78)

By using equalities (72), from Eq. (78), we have

\[ \sum_{p=0}^8 c_{p11} \varphi_p(x) - \frac{1}{\lambda' + 2 \mu} Y''_{11}(x) = \sum_{p=1}^8 \lambda^2 c_{p11} + \frac{1}{\lambda' + 2 \mu} Y''_{11}(x) \]  

(79)

Also, we have

\[ \frac{1}{\rho \omega^2} \varphi_0(x) + \frac{1}{\mu} Y''_{11}(x) = \frac{1}{\rho \omega^2} \left[ Y''_{11}(x) - \frac{i \lambda}{4\pi} + \varphi_{pp}(x) \right] + \frac{1}{\mu} Y''_{11}(x) = \frac{1}{\rho \omega^2} \left[ Y''_{11}(x) - \frac{i \lambda}{4\pi} + \varphi_{pp}(x) \right] \]  

(80)

Taking into account (77), (79) and (80) and that \( \varDelta Y''_{11}(x) = 0(x \neq 0) \), from Eq. (75), we have

\[ G_y(x) - G'_y(x) = -\frac{1}{\rho \omega^2} \left[ \frac{\partial^2}{\partial x_i \partial x_j} - R'_{ij} \right] Y''_{11}(x) + \frac{\partial^2}{\partial x_i \partial x_j} \sum_{p=1}^8 c_{p11} \varphi_{pp}(x) + \frac{1}{\rho \omega^2} R'_{ij} \varphi_{pp}(x) \]

\[ = \frac{1}{\rho \omega^2} \varDelta \delta_{ij} Y''_{11}(x) + \text{constant} + O(|x|) = \text{constant} + O(|x|) \]  

\( i, j = 1, 2, 3 \)

Similarly, other formulae of (74) can be proved.
Therefore, matrix $G'(x)$ gives the singular part of the fundamental solution $G(x)$ in the neighborhood of the origin.

### 6 PARTICULAR CASES

1. If we put $\omega = 0$, that is, taking static case in the Eqs. (26)-(30), we can obtain the fundamental solution of partial differential equations in the generalized theory of thermoelastic diffusion materials with double porosity in case of equilibrium oscillations in terms of elementary functions. In this case, operator $\Gamma(\Delta)$, vector $\Psi(x)$ and the matrix operators $\Theta(\Delta)$, $R(D_x)$ and $Y(x) = \|Y_g(x)\|_{b_7}$ are changed in the following forms:

   **i.** $\Gamma(\Delta) = \Delta^2 + \lambda^2$, where $\lambda^2, i = 1, 2$ are the roots of the equation $\det N(-\kappa) = 0$, w.r.t. $\kappa$ and

   $N'(\Delta) = \begin{vmatrix} \lambda + 2\mu & -g_1 & -g_2 & 0 & 0 \\ g_1 & t_1\Delta - d_1 & r_1\Delta - e_1 & 0 & 0 \\ g_2 & r_1\Delta - e_1 & t_2\Delta - f & 0 & 0 \\ -s_1 & \xi_{11} & \xi_{22} & K & 0 \\ -l_1 & w & \nu & 0 & D \end{vmatrix}_{b_5}$

   **ii.** $\Psi(x) = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5)$, where

   $\Psi_1 = \frac{1}{\mu} \Gamma(\Delta) J + w_{11}(\Delta) \text{grad div } H + w_{21}(\Delta) \text{grad } L + w_{31}(\Delta) \text{grad } M$,

   $\Psi_2 = w_{12}(\Delta) \text{div } H + w_{22}(\Delta) L + w_{32}(\Delta) M$,

   $\Psi_3 = w_{13}(\Delta) \text{div } H + w_{23}(\Delta) L + w_{33}(\Delta) M$,

   $\Psi_4 = w_{14}(\Delta) \text{div } H + w_{24}(\Delta) L + w_{34}(\Delta) M + w_{44}(\Delta) Z$,

   $\Psi_5 = w_{15}(\Delta) \text{div } H + w_{25}(\Delta) L + w_{35}(\Delta) M + w_{55}(\Delta) X$.

   **iii.** $\Theta(\Delta) = \|\Theta_{pq}(\Delta)\|_{b_7}$, where

   $\Theta_{pq}(\Delta) = \Gamma(\Delta), \Theta_{pq}(\Delta) = \Gamma(\Delta), \Theta_{pq}(\Delta) = 0, i = 1, 2, 3, j = 4, ..., 7, p, q = 1, ..., 7, p \neq q$

   **iv.** $R(D_x) = \|R_{pq}(D_x)\|_{b_7}$, where

   $R_{pq}(D_x) = \frac{1}{\mu} \Gamma(\Delta) \delta_{q} + w_{11}(\Delta) \frac{\partial^2}{\partial x_j \partial x_i}, R_{pq+2}(D_x) = w_{1p}(\Delta) \frac{\partial}{\partial x_j}, R_{pq+2}(D_x) = w_{e1}(\Delta) \frac{\partial}{\partial x_j}$,

   $R_{q0}(D_x) = R_{2q}(D_x) = R_{3q}(D_x) = R_{4q}(D_x) = 0$,

   $R_{e+,p+2}(D_x) = w_{ep}(\Delta), R_{e+k}(D_x) = w_{e-2k-2}(\Delta)$

   $i, j = 1, 2, 3, p = 2, 3, 4, 5, e = 2, 3, k = 6, 7, q = 1, ..., 5$.

   **v.** $Y(x) = \|Y_g(x)\|_{b_7}$, where

   $Y_{pq}(x) = r_{1q} s_{1p}(x) + r_{2q} s_{2p}(x) + \sum_{g=1}^{2} r_{1qg+2} s_{g+p}(x), Y_{pq}(x) = r_{1q} s_{1p}(x) + \sum_{g=1}^{2} r_{2qg+2} s_{g+p}(x)$,

   $Y_{pq}(x) = 0, p = 1, 2, 3, e = 4, 5, 6, 7, q, h = 1, ..., 7, q \neq h$
Here
\[ s_1^* (x) = - \frac{1}{4\pi |x|} \cdot s_2^* (x) = \frac{|x|}{8\pi} r_{11} = - \frac{(\lambda_1^3 + \lambda_2^3)(\lambda_1^4 + \lambda_2^4)}{\lambda_1^2 \lambda_2^8}, \]
\[ r_{12} = r_{23} = r_{24} = \frac{(\lambda_1^3 + \lambda_2^3 + \lambda_1 \lambda_2)(\lambda_1^3 + \lambda_2^3)}{\lambda_1^2 \lambda_2^6 (\lambda_1^2 + \lambda_2^2)}, \]
\[ r_{13} = \frac{1}{\lambda_1^2 (\lambda_1^2 - \lambda_2^2)}, r_{14} = \frac{1}{\lambda_2^2 (\lambda_1^2 - \lambda_2^2)}, r_{23} = - \frac{1}{\lambda_1^2 (\lambda_1^2 - \lambda_2^2)}, r_{24} = - \frac{1}{\lambda_2^2 (\lambda_1^2 - \lambda_2^2)}. \]

2. If double porosity effect is skipped, then we can construct the fundamental solution of a system of equations in the generalized theory of thermoelastic diffusion as given by Kumar and Kansal [19].
3. If the diffusion effect is neglected, then the fundamental solution of partial differential equations in the generalized theory of thermoelastic materials with double porosity can be obtained similarly given by Scarpetta et al. [17].
4. If further thermal effect is omitted, then we can obtain the fundamental solution of partial differential equations in the theory of elastic materials with double porosity similarly given by Svanadze and de Cicco [15].

7 CONCLUSIONS

The fundamental solution of a system of equations in the generalized theory of thermoelastic diffusion materials with double porosity in case of steady oscillations in terms of elementary functions has been constructed. The fundamental solution makes it possible to investigate three-dimensional boundary value problems of generalized theories of thermoelastic diffusion with double porosity by potential method [20].

REFERENCES


