Journal of Solid Mechanics Vol. 11, No. 2 (2019) pp. 281-296
DOI: 10.22034/JSM.2019.665384

Fundamental Solution in the Theory of Thermoelastic Diffusion Materials with Double Porosity

T. Kansal *

Department of Mathematics, M.N.College, Shahabad, 136135, India

Received 9 February 2019; accepted 29 March 2019

ABSTRACT

The main purpose of present article is to find the fundamental solution of partial differential equations in the generalized theory of thermoelastic diffusion materials with double porosity in case of steady oscillations in terms of elementary functions.

Keywords : Thermoelastic; Diffusion; Double porosity; Steady oscillations.

1 INTRODUCTION

LORD and Shulman [1] established the theory of generalized thermoelasticity by modifying Fourier's law of heat conduction. This theory overcomes the shortcomings of classical theory of thermoelasticity in which thermal waves propagate with infinite velocity. The transfer of mass of a substance from the high concentration regions to low concentration regions is known as diffusion. Nowacki [2-5] developed the classical thermoelasticity with mass diffusion. With the help of modified Fourier's and Fick's laws, Sherief et al. [6] established generalized theory of thermoelasticity with mass diffusion. Iesan [7] constructed the linear theory of thermolastic materials with single voids. Aouadi [8] developed a theory of thermoelastic diffusion materials with voids and derived various theorems. The double porosity model represents a double porous structure, one is macro porosity which is connected to pores and other is micro porosity which is connected to fissures. Barenblatt et al. [9] and Warren and Root [10] extended Darcy's law to describe fluid flow through undeformable double porosity materials. Wilson and Aifantis [11] developed the theory for deformable materials with double porosity. Iesan and Quintanilla [12] derived a non-linear theory of thermoelastic solids with double porosity structure based upon Nunziato-Cowin theory of materials with voids. This theory was not based upon Darcy's law. Kansal [13] established linear generalized theory of thermoelastic diffusion with double porosity. The construction of fundamental solution of a system of partial differential equations and establishment of their basic properties are required to study the boundary value problems by using potential method. The concept of fundamental solutions has great importance in many mathematical, mechanical, physical, and engineering applications. For example, the application of fundamental solutions to a recently developed area of boundary element methods has provided a concrete advantage in that an integral representation of the solution to a boundary value problem in terms of a fundamental solution can be solved more easily by numerical methods with respect to a differential equation with some specified boundary and initial conditions. Several authors [14-18] constructed the fundamental solutions in the theories of elasticity and thermoelasticity for materials with double porosity.

*Corresponding author. Tel.: +91 9253010657.
E-mail address: tarun1_kansal@yahoo.co.in (T. Kansal).

In this paper, firstly the basic equations for homogeneous anisotropic generalized thermoelastic diffusion solid with double voids are considered. After reducing to the isotropic case and assuming the solutions in case of steady oscillations, the fundamental solutions of the governing equations are constructed in terms of elementary functions. Some basic properties of fundamental matrix are also discussed. Finally, some particular cases are obtained.

2 BASIC EQUATIONS

Let $\mathbf{x}=(x_1,x_2,x_3)$ be the point of the Euclidean three-dimensional space $E^3, |\mathbf{x}|=\left(x_1^2+x_2^2+x_3^2\right)^{1/2}, \mathbf{D_x}=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$ and let t denote the time variable. Following Kansal [13], the governing equations for an anisotropic homogeneous generalized thermoelastic diffusion solid with double porosity in the absence of body forces, heat and mass diffusive sources are

Constitutive relations:

$$\sigma_{ij}=c_{ijkl}e_{kl}+p_{ij}\phi+\gamma_{ij}\psi-a_{ij}\theta-b_{ij}C \tag{1}$$

$$\Omega_i=q_{ij}\phi,_j+\alpha_{ij}\psi,_j \tag{2}$$

$$\chi_i=\alpha_{ij}\phi,_j+f_{ij}\psi,_j \tag{3}$$

$$\xi=-p_{ij}e_{ij}-d^*\phi-\alpha_1\psi+\gamma_1\theta+vC \tag{4}$$

$$\zeta=-\gamma_{ij}e_{ij}-\alpha_1\phi-f\,\psi+\gamma_2\theta+mC \tag{5}$$

$$\rho S=a_{ij}e_{ij}+\gamma_1\phi+\gamma_2\psi+\frac{\rho C_e\theta}{T_0}++aC \tag{6}$$

$$\rho T_0\dot{S}=-q_{i,i} \tag{7}$$

$$-\eta_{i,i}=\dot{C} \tag{8}$$

$$P=-b_{ij}e_{ij}-v\,\phi-m\psi-a\theta+bC \tag{9}$$

Equations of motion:

$$\sigma_{ij,j}=\rho\ddot{u}_i \tag{10}$$

Balance of equilibrated forces:

$$\Omega_{i,i}+\xi=k_1\ddot{\phi} \tag{11}$$

$$\chi_{i,i}+\zeta=k_2\ddot{\psi} \tag{12}$$

Equation of heat conduction:

$$q_i + \tau_0 \dot{q}_i = -K_{ij}\theta,_j \tag{13}$$

Equation of chemical potential:

$$\eta_i + \tau^0 \dot{\eta}_i = -d_{ij} P_{,j} \tag{14}$$

In the equations mentioned above, $\sigma_{ij}, e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are, respectively, the components of the stress and strain tensor, u_i are the components of displacement vector $\mathbf{u}$, Ω_i, χ_i are equilibrated stress vectors, ξ, ζ are the intrinsic equilibrated body forces associated to macro pores and fissures respectively, ρ is the density, C_e is the specific heat at the constant strain, q_i, η_i are the components of heat and mass diffusion flux vectors $\mathbf{q}, \boldsymbol{\eta}$ respectively, k_1, k_2 are the coefficients of equilibrated inertia, T_0 is the absolute temperature in the reference state, θ is the temperature variation from the absolute temperature T_0, C is the concentration of the diffusion material in the elastic body, *S*, *P* are entropy and chemical potential per unit mass respectively, ϕ, ψ are change in volume fraction fields from the reference volume fraction, $c_{ijkl}(= c_{klij} = c_{jikl} = c_{ijlk})$ is the tensor of elastic constants, $K_{ij}(= K_{ji}), d_{ij}(= d_{ji})$ are respectively the components of thermal conductivity and diffusivity, a_{ij}, b_{ij} are tensors of thermal and diffusion moduli respectively, *a*, *b* are, respectively, the coefficients describing the measure of thermodiffusion and of mass diffusion effects, τ_0 is the thermal relaxation time which ensures that the heat conduction equation predicts finite speeds of heat propagation and τ^0 is the diffusion relaxation time, which ensures that the equation satisfied by the concentration also predicts finite speeds of propagation of matter from one medium to the other.

If we put $c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \mu \delta_{il}\delta_{jk}, a_{ij} = \xi_1 \delta_{ij}, b_{ij} = \xi_2 \delta_{ij}, p_{ij} = p_1 \delta_{ij}, \gamma_{ij} = p_2 \delta_{ij}$, $q_{ij} = t_1 \delta_{ij}, \alpha_{ij} = r_1 \delta_{ij}$, $f_{ij} = t_2 \delta_{ij}$, $K_{ij} = K\delta_{ij}, d_{ij} = D\delta_{ij}$ in the above equations, then from Eqs. (10)-(14) with the aid of Eqs. (1)-(9), we obtain the basic equations for homogeneous isotropic thermoelastic diffusion material with double porosity as:

$$\mu \Delta \mathbf{u} + (\lambda + \mu)\,\text{grad div}\,\mathbf{u} + p_1\, grad\,\phi + p_2\, grad\,\psi - \xi_1\, grad\,\theta - \xi_2\, grad\,C = \rho \ddot{\mathbf{u}} \tag{15}$$

$$-p_1\, div\,\mathbf{u} + (t_1 \Delta - d^*)\phi + (r_1 \Delta - \alpha_1)\psi + \gamma_1 \theta + vC = k_1 \ddot{\phi} \tag{16}$$

$$-p_2\, div\,\mathbf{u} + (r_1 \Delta - \alpha_1)\phi + (t_2 \Delta - f)\psi + \gamma_2 \theta + mC = k_2 \ddot{\psi} \tag{17}$$

$$\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\left[\rho C_e \theta + T_0(\xi_1\, div\mathbf{u} + \gamma_1 \phi + \gamma_2 \psi + aC)\right] = K\Delta\theta \tag{18}$$

$$D\Delta\left[-\xi_2\, div\mathbf{u} - v\phi - m\psi - a\theta + bC\right] = \left(\frac{\partial}{\partial t} + \tau^0 \frac{\partial^2}{\partial t^2}\right)C \tag{19}$$

In the upcoming sections, the chemical potential has been used as a state variable instead of the concentration. The Eqs. (15)-(19) can be rewritten as:

$$\mu \Delta \mathbf{u} + (\lambda' + \mu)\,\text{grad div}\,\mathbf{u} + g_1\, grad\,\phi + g_2\, grad\,\psi - s_1\, grad\,\theta - l_1\, grad\,P = \rho \ddot{\mathbf{u}} \tag{20}$$

$$-g_1\, div\mathbf{u} + (t_1 \Delta - d_1)\phi + (r_1 \Delta - \varepsilon_{11})\psi + \xi_{11}\theta + wP = k_1 \ddot{\phi} \tag{21}$$

$$-g_2\, div\mathbf{u} + (r_1 \Delta - \varepsilon_{11})\phi + (t_2 \Delta - f_1)\psi + \xi_{22}\theta + vP = k_2 \ddot{\psi} \tag{22}$$

$$-\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)T_0\left[s_1\, div\,\mathbf{u}+\xi_{11}\phi+\xi_{22}\psi+c^*\theta+sP\right]+K\,\Delta\theta=0 \tag{23}$$

$$-\left(\frac{\partial}{\partial t}+\tau^0\frac{\partial^2}{\partial t^2}\right)\left[l_1\, div\,\mathbf{u}+w\phi+v\psi+s\theta+nP\right]+D\,\Delta P=0 \tag{24}$$

where

$$n=\frac{1}{b},l_1=n\xi_2,s_1=\xi_1+al_1,g_1=p_1-vl_1,g_2=p_2-ml_1,\lambda'=\lambda-l_1\xi_2,s=an,v=mn,w=vn,$$
$$d_1=d^*-vw,\varepsilon_{11}=\alpha_1-vv,\xi_{11}=\gamma_1+vs,f_1=f-mv,\xi_{22}=\gamma_2+ms,c^*=\frac{\rho C_e}{T_0}+as.$$

3 STEADY OACILLATIONS

Now, we consider the case of steady oscillations. We assume the displacement vector, volume fraction fields, temperature change and chemical potential functions as:

$$[\mathbf{u}(\mathbf{x},t),\ \phi(\mathbf{x},t),\ \psi(\mathbf{x},t),\ \theta(\mathbf{x},t),\ P(\mathbf{x},t)]=\mathrm{Re}[(\mathbf{u}^*,\phi^*,\psi^*,\theta^*,P^*)e^{-\iota\omega t}] \tag{25}$$

where, ω is oscillation frequency.

Using Eq. (25) in Eqs. (20)–(24) and omitting asterisk (*) for simplicity, we obtain the system of equations of steady oscillations as:

$$[\mu\Delta+(\lambda'+\mu)\text{ grad div}+\rho\omega^2]\mathbf{u}+g_1\, grad\,\phi+g_2\, grad\,\psi-s_1\, grad\,\theta-l_1\, grad\,P=\mathbf{0} \tag{26}$$

$$-g_1\, div\,\mathbf{u}+[t_1\,\Delta-d_1+k_1\omega^2]\phi+(r_1\,\Delta-\varepsilon_{11})\psi+\xi_{11}\theta+wP=0 \tag{27}$$

$$-g_2\, div\,\mathbf{u}+(r_1\,\Delta-\varepsilon_{11})\phi+[t_2\,\Delta-f_1+k_2\omega^2]\psi+\xi_{22}\theta+vP=0 \tag{28}$$

$$\tau_1T_0[s_1\, div\,\mathbf{u}+\xi_{11}\phi+\xi_{22}\psi]+[K\,\Delta+\tau_1T_0c^*]\theta+\tau_1T_0sP=0 \tag{29}$$

$$\tau^1[l_1\, div\,\mathbf{u}+w\phi+v\psi+s\theta]+[D\,\Delta+\tau^1n]P=0 \tag{30}$$

where, $\tau_1=\iota\omega(1-\iota\omega\tau_0),\tau^1=\iota\omega(1-\iota\omega\tau^0)$.

We introduce the second order matrix differential operators with constant coefficients

1) $\mathbf{F}(\mathbf{D_x})=\left\|F_{gh}(\mathbf{D_x})\right\|_{7\times7}$

where

$$F_{pq}(\mathbf{D_x})=[\mu\Delta+\rho\omega^2]\delta_{pq}+(\lambda'+\mu)\frac{\partial^2}{\partial x_p\partial x_q},F_{p4}(\mathbf{D_x})=g_1\frac{\partial}{\partial x_p},F_{p5}(\mathbf{D_x})=g_2\frac{\partial}{\partial x_p},F_{p6}(\mathbf{D_x})=-s_1\frac{\partial}{\partial x_p},F_{p7}(\mathbf{D_x})=-l_1\frac{\partial}{\partial x_p},$$
$$F_{4q}(\mathbf{D_x})=-g_1\frac{\partial}{\partial x_q},F_{44}(\mathbf{D_x})=t_1\Delta-d_1+k_1\omega^2,F_{45}(\mathbf{D_x})=F_{54}(\mathbf{D_x})=r_1\Delta-\varepsilon_{11},F_{46}(\mathbf{D_x})=\xi_{11},F_{47}(\mathbf{D_x})=w,F_{5q}(\mathbf{D_x})=-g_2\frac{\partial}{\partial x_q},$$

$$F_{55}(\mathbf{D_x}) = t_2\Delta - f_1 + k_2\omega^2, F_{56}(\mathbf{D_x}) = \xi_{22}, F_{57}(\mathbf{D_x}) = \nu, F_{6q}(\mathbf{D_x}) = \tau_1 s_1 T_0 \frac{\partial}{\partial x_q}, F_{64}(\mathbf{D_x}) = \tau_1\xi_{11}T_0, F_{65}(\mathbf{D_x}) = \tau_1\xi_{22}T_0,$$

$$F_{66}(\mathbf{D_x}) = K\Delta + \tau_1 c^* T_0, F_{67}(\mathbf{D_x}) = \tau_1 s T_0, F_{7q}(\mathbf{D_x}) = \tau^1 l_1 \frac{\partial}{\partial x_q}, F_{74}(\mathbf{D_x}) = \tau^1 w, F_{75}(\mathbf{D_x}) = \tau^1\nu, F_{76}(\mathbf{D_x}) = \tau^1 s, F_{77}(\mathbf{D_x}) = \tau^1 n + D\Delta \quad p,q = 1,2,3.$$

2) $\mathbf{F}^*(\mathbf{D_x}) = \left\| F^*_{gh}(\mathbf{D_x}) \right\|_{7\times 7}$

where

$$F^*_{pq}(\mathbf{D_x}) = \mu\Delta\delta_{pq} + (\lambda' + \mu)\frac{\partial^2}{\partial x_p \partial x_q}, F^*_{pi}(\mathbf{D_x}) = 0, F^*_{ip}(\mathbf{D_x}) = 0, F^*_{44}(\mathbf{D_x}) = t_1\Delta, F^*_{45}(\mathbf{D_x}) = F^*_{54}(\mathbf{D_x}) = r_1\Delta, F^*_{ej}(\mathbf{D_x}) = F^*_{je}(\mathbf{D_x}) = 0,$$

$$F^*_{55}(\mathbf{D_x}) = t_2\Delta, F^*_{66}(\mathbf{D_x}) = K\Delta, F^*_{67}(\mathbf{D_x}) = F^*_{76}(\mathbf{D_x}) = 0, F^*_{77}(\mathbf{D_x}) = D\Delta \quad p,q = 1,2,3 \quad i = 4,.....,7 \quad e = 4,5 \quad j = 6,7.$$

The system of Eqs. (26)–(30) can be written as:

$$\mathbf{F}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$$

where $\mathbf{U} = (\mathbf{u}, \phi, \psi, \theta, P)$ is a seven-component vector function for E^3. The matrix $\mathbf{F}^*(\mathbf{D_x})$ is called the principal part of the operator $\mathbf{F}(\mathbf{D_x})$.

Definition 1: The operator $\mathbf{F}(\mathbf{D_x})$ is said to be elliptic if $\det \mathbf{F}^*(\boldsymbol{\kappa}) \neq 0$, where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)$.

We have, $\det \mathbf{F}^*(\boldsymbol{\kappa})$ = det

$$\begin{pmatrix} \mu|\boldsymbol{\kappa}|^2 + (\lambda'+\mu)\kappa_1^2 & (\lambda'+\mu)\kappa_1\kappa_2 & (\lambda'+\mu)\kappa_1\kappa_3 & 0 & 0 & 0 & 0 \\ (\lambda'+\mu)\kappa_1\kappa_2 & \mu|\boldsymbol{\kappa}|^2 + (\lambda'+\mu)\kappa_2^2 & (\lambda'+\mu)\kappa_2\kappa_3 & 0 & 0 & 0 & 0 \\ (\lambda'+\mu)\kappa_1\kappa_3 & (\lambda'+\mu)\kappa_2\kappa_3 & \mu|\boldsymbol{\kappa}|^2 + (\lambda'+\mu)\kappa_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_1|\boldsymbol{\kappa}|^2 & r_1|\boldsymbol{\kappa}|^2 & 0 & 0 \\ 0 & 0 & 0 & r_1|\boldsymbol{\kappa}|^2 & t_2|\boldsymbol{\kappa}|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K|\boldsymbol{\kappa}|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D|\boldsymbol{\kappa}|^2 \end{pmatrix}$$

$$= \mu^2(\lambda' + 2\mu)KD(t_1t_2 - r_1^2)|\boldsymbol{\kappa}|^{14}$$

Therefore, operator $\mathbf{F}(\mathbf{D_x})$ is an elliptic differential operator if and only if

$$\mu(\lambda' + 2\mu)KD(t_1t_2 - r_1^2) \neq 0 \tag{31}$$

Definition 2: The fundamental solution of the system of Eqs. (26)-(30) (the fundamental matrix of operator $\mathbf{F}$) is the matrix $\mathbf{G}(\mathbf{x}) = \left\| G_{gh}(\mathbf{x}) \right\|_{7\times 7}$ satisfying condition

$$\mathbf{F}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}) \tag{32}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{I}(\mathbf{x}) = \left\| \delta_{gh} \right\|_{7\times 7}$ is the unit matrix and $\mathbf{x} \in E^3$. Now, we construct $\mathbf{G}(\mathbf{x})$ in terms of elementary functions.

4 FUNDAMENTAL SOLUTION OF A SYSTEM OF EQUATIONS OF STEADY OSCILLATIONS

Let us consider the system of non-homogeneous equations

$$[\mu\Delta+(\lambda'+\mu)\,\text{grad div}+\rho\omega^2]\,\mathbf{u}-g_1\,grad\,\phi-g_2\,grad\,\psi+\tau_1T_0s_1\,grad\,\theta+\tau^1l_1\,grad\,P=\mathbf{H} \tag{33}$$

$$g_1\,div\,\mathbf{u}+[t_1\,\Delta-d_1+k_1\omega^2]\phi+(r_1\,\Delta-\varepsilon_{11})\psi+\tau_1T_0\xi_{11}\theta+\tau^1wP=L \tag{34}$$

$$g_2\,div\,\mathbf{u}+(r_1\,\Delta-\varepsilon_{11})\phi+[t_2\,\Delta-f_1+k_2\omega^2]\psi+\tau_1T_0\xi_{22}\theta+\tau^1\nu P=M \tag{35}$$

$$-s_1\,div\,\mathbf{u}+\xi_{11}\phi+\xi_{22}\psi+[K\Delta+\tau_1T_0c^*]\theta+\tau^1sP=Z \tag{36}$$

$$-l_1\,div\,\mathbf{u}+w\phi+\nu\psi+\tau_1T_0s\theta+[D\Delta+\tau^1n]P=X \tag{37}$$

where $\mathbf{H}$ is three-component vector function on E^3; L, M, Z and X are scalar functions on E^3.

The system of Eqs. (33)–(37) may be written in the form

$$\mathbf{F^{tr}}(\mathbf{D_x})\mathbf{U}(\mathbf{x})=\mathbf{Q}(\mathbf{x}) \tag{38}$$

where $\mathbf{F^{tr}}$ is the transpose of matrix $\mathbf{F}$, $\mathbf{Q}=(\mathbf{H},L,M,Z,X)$, $\mathbf{x}\in E^3$.

Applying the operator div to Eq. (33), we obtain

$$[(\lambda'+2\mu)\,\Delta+\rho\omega^2]\text{div}\,\mathbf{u}-g_1\,\Delta\phi-g_2\,\Delta\psi+\tau_1T_0s_1\,\Delta\theta+\tau^1l_1\,\Delta P=div\,\mathbf{H} \tag{39}$$

The Eqs. (34)–(37) and (39) may be written in the form

$$\mathbf{N}(\Delta)\mathbf{S}=\bar{\mathbf{Q}} \tag{40}$$

where $\mathbf{S}=(div\,\mathbf{u},\phi,\psi,\theta,P),\bar{\mathbf{Q}}=(w_1,w_2,w_3,w_4,w_5)=(div\,\mathbf{H},L,M,Z,X)$ and

$$\mathbf{N}(\Delta)=\|N_{gh}(\Delta)\|_{5\times5}=\begin{Vmatrix}(\lambda'+2\mu)\Delta+\rho\omega^2 & -g_1\Delta & -g_2\Delta & \tau_1T_0s_1\Delta & \tau^1l_1\Delta\\ g_1 & t_1\Delta-d_1+k_1\omega^2 & r_1\Delta-\varepsilon_{11} & \tau_1T_0\xi_{11} & \tau^1w\\ g_2 & r_1\Delta-\varepsilon_{11} & t_2\Delta-f_1+k_2\omega^2 & \tau_1T_0\xi_{22} & \tau^1\nu\\ -s_1 & \xi_{11} & \xi_{22} & K\Delta+\tau_1T_0c^* & \tau^1s\\ -l_1 & w & \nu & \tau_1T_0s & D\Delta+\tau^1n\end{Vmatrix}_{5\times5} \tag{41}$$

Eqs. (34)–(37) and (39) may be also written as:

$$\Gamma(\Delta)\mathbf{S}=\mathbf{\Psi} \tag{42}$$

where

$$\mathbf{\Psi}=(\Psi_1,\Psi_2,\Psi_3,\Psi_4,\Psi_5),\ \Psi_p=\frac{1}{M^*}\sum_{i=1}^{5}N^*_{ip}w_i,\Gamma(\Delta)=\frac{\det\mathbf{N}(\Delta)}{M^*},$$
$$M^*=(\lambda'+2\mu)KD(t_1t_2-r_1^2)\ \ p=1,...,5. \tag{43}$$

and N^*_{ip} is the cofactor of the element N_{ip} of the matrix $\mathbf{N}$. From Eqs. (41) and (43), we see that

$$\Gamma(\Delta)=\prod_{i=1}^{5}(\Delta+\lambda_i^2)$$

where $\lambda_i^2, i=1,.....,5$ are the roots of the equation $\Gamma(-\kappa)=0$, w.r.t. κ.

Applying operator $\Gamma(\Delta)$ to Eq. (33), we get

$$\Gamma(\Delta)(\Delta+\lambda_6^2)\mathbf{u}=\mathbf{\Psi}' \tag{44}$$

where $\lambda_6^2=\dfrac{\rho\omega^2}{\mu}$ and

$$\mathbf{\Psi}'=\frac{1}{\mu}\Big[\Gamma(\Delta)\mathbf{H}-grad\big[(\lambda'+\mu)\Psi_1-g_1\Psi_2-g_2\Psi_3+\tau_1T_0s_1\Psi_4+\tau^1l_1\Psi_5\big]\Big] \tag{45}$$

From Eqs. (42) and (44), we have

$$\mathbf{\Theta}(\Delta)\mathbf{U}(\mathbf{x})=\hat{\mathbf{\Psi}}(\mathbf{x}) \tag{46}$$

where

$$\hat{\mathbf{\Psi}}(\mathbf{x})=(\mathbf{\Psi}',\Psi_2,\Psi_3,\Psi_4,\Psi_5) \quad \text{and} \quad \mathbf{\Theta}(\Delta)=\left\|\Theta_{pq}(\Delta)\right\|_{7\times7}$$
$$\Theta_{ii}(\Delta)=\Gamma(\Delta)(\Delta+\lambda_6^2),\ \Theta_{jj}(\Delta)=\Gamma(\Delta),\Theta_{pq}(\Delta)=0\ \ i=1,2,3\ \ j=4,.....,7\ \ p,q=1,....,7\ \ p\neq q$$

Eqs. (43) and (45) can be rewritten in the form

$$\begin{aligned}
\mathbf{\Psi}'&=\left[\frac{1}{\mu}\Gamma(\Delta)\mathbf{J}+w_{11}(\Delta)\,grad\,div\right]\mathbf{H}+w_{21}(\Delta)\,\mathrm{grad}\,\mathrm{L}+w_{31}(\Delta)\,grad\,M+w_{41}(\Delta)\,grad\,Z+w_{51}(\Delta)\,grad\,X,\\
\Psi_2&=w_{12}(\Delta)\,div\,\mathbf{H}+w_{22}(\Delta)L+w_{32}(\Delta)M+w_{42}(\Delta)Z+w_{52}(\Delta)X,\\
\Psi_3&=w_{13}(\Delta)\,div\,\mathbf{H}+w_{23}(\Delta)L+w_{33}(\Delta)M+w_{43}(\Delta)Z+w_{53}(\Delta)X,\\
\Psi_4&=w_{14}(\Delta)\,div\,\mathbf{H}+w_{24}(\Delta)L+w_{34}(\Delta)M+w_{44}(\Delta)Z+w_{54}(\Delta)X,\\
\Psi_5&=w_{15}(\Delta)\,div\,\mathbf{H}+w_{25}(\Delta)L+w_{35}(\Delta)M+w_{45}(\Delta)Z+w_{55}(\Delta)X,
\end{aligned} \tag{47}$$

where $\mathbf{J}=\left\|\delta_{pq}\right\|_{3\times3}$ is the unit matrix.

In the above equation, the following notations have been used:

$$w_{p1}(\Delta)=-\frac{1}{\mu M^*}\Big[(\lambda'+\mu)N_{p1}^*(\Delta)-g_1N_{p2}^*(\Delta)-g_2N_{p3}^*(\Delta)+\tau_1T_0s_1N_{p4}^*(\Delta)+\tau^1l_1N_{p5}^*(\Delta)\Big],$$
$$w_{pj}(\Delta)=\frac{N_{pj}^*(\Delta)}{M^*}\quad p=1,.....,5\ \ j=2,.....,5.$$

From Eq. (47), we have

$$\hat{\mathbf{\Psi}}(\mathbf{x})=\mathbf{R}^{\mathbf{tr}}(\mathbf{D_x})\mathbf{Q}(\mathbf{x}) \tag{48}$$

where $\mathbf{R}(\mathbf{D_x})=\left\|R_{gh}(\mathbf{D_x})\right\|_{7\times7}$,

$$R_{ij}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma(\Delta)\delta_{ij} + w_{11}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j}, R_{i;p+2}(\mathbf{D_x}) = w_{1p}(\Delta)\frac{\partial}{\partial x_i}, R_{p+2;i}(\mathbf{D_x}) = w_{p1}(\Delta)\frac{\partial}{\partial x_i},$$
$$R_{p+2;l+2}(\mathbf{D_x}) = w_{pl}(\Delta)\ i,j = 1,2,3\ p,l = 2,3,4,5. \tag{49}$$

From Eqs. (38), (46) and (48), we obtain $\mathbf{\Theta U} = \mathbf{R^{tr}Q} = \mathbf{R^{tr}F^{tr}U}$. It implies that

$$\mathbf{R^{tr}F^{tr}} = \mathbf{\Theta},\ \mathbf{F(D_x)R(D_x)} = \mathbf{\Theta}(\Delta) \tag{50}$$

We assume that $\lambda_p^2 \neq \lambda_q^2 \neq 0\ p,q = 1,.....,6$ and $M^*\mu \neq 0$. Let

$$\mathbf{Y(x)} = \left\| Y_{ij}(\mathbf{x}) \right\|_{7\times 7}, Y_{pp}(\mathbf{x}) = \sum_{g=1}^{6} r_{1g}\varsigma_g(\mathbf{x}), Y_{ee}(\mathbf{x}) = \sum_{g=1}^{5} r_{2g}\varsigma_g(\mathbf{x}),$$
$$Y_{qh}(\mathbf{x}) = 0\ p = 1,2,3\ e = 4,......,7\ q,h = 1,........,7\ q \neq h \tag{51}$$

where

$$\varsigma_g(\mathbf{x}) = -\frac{e^{\iota\lambda_g|\mathbf{x}|}}{4\pi|\mathbf{x}|}, r_{1g} = \prod_{i=1,i\neq g}^{6}(\lambda_i^2 - \lambda_g^2)^{-1}, r_{2h} = \prod_{i=1,i\neq h}^{5}(\lambda_i^2 - \lambda_h^2)^{-1}\ g = 1,...,6\ h = 1,....,5. \tag{52}$$

Lemma 1: The matrix $\mathbf{Y}$ is the fundamental matrix of the operator $\mathbf{\Theta}(\Delta)$ i.e.

$$\mathbf{\Theta}(\Delta)\mathbf{Y(x)} = \delta(\mathbf{x})\mathbf{I(x)} \tag{53}$$

To prove the lemma, it is sufficient to prove that

$$\Gamma(\Delta)(\Delta + \lambda_6^2)Y_{11}(\mathbf{x}) = \delta(\mathbf{x}),\ \Gamma(\Delta)Y_{44}(\mathbf{x}) = \delta(\mathbf{x}) \tag{54}$$

Consider $\sum_{i=1}^{6} r_{1i} = \frac{1}{z_7}\sum_{j=1}^{6}(-1)^j z_j$, where

$$z_1 = \prod_{i=3}^{6}(\lambda_2^2 - \lambda_i^2)\prod_{j=4}^{6}(\lambda_3^2 - \lambda_j^2)\prod_{p=5}^{6}(\lambda_4^2 - \lambda_p^2)(\lambda_5^2 - \lambda_6^2), z_2 = \prod_{i=3}^{6}(\lambda_1^2 - \lambda_i^2)\prod_{j=4}^{6}(\lambda_3^2 - \lambda_j^2)\prod_{p=5}^{6}(\lambda_4^2 - \lambda_p^2)(\lambda_5^2 - \lambda_6^2),$$
$$z_3 = \prod_{\substack{i=2\\ i\neq 3}}^{6}(\lambda_1^2 - \lambda_i^2)\prod_{j=4}^{6}(\lambda_2^2 - \lambda_j^2)\prod_{p=5}^{6}(\lambda_4^2 - \lambda_p^2)(\lambda_5^2 - \lambda_6^2), z_4 = \prod_{\substack{i=2\\ i\neq 4}}^{6}(\lambda_1^2 - \lambda_i^2)\prod_{\substack{j=3\\ j\neq 4}}^{6}(\lambda_2^2 - \lambda_j^2)\prod_{p=5}^{6}(\lambda_3^2 - \lambda_p^2)(\lambda_5^2 - \lambda_6^2),$$
$$z_5 = \prod_{\substack{i=2\\ i\neq 5}}^{6}(\lambda_1^2 - \lambda_i^2)\prod_{\substack{j=3\\ j\neq 5}}^{6}(\lambda_2^2 - \lambda_j^2)\prod_{\substack{p=4\\ p\neq 5}}^{6}(\lambda_3^2 - \lambda_p^2)(\lambda_4^2 - \lambda_6^2), z_6 = \prod_{i=2}^{5}(\lambda_1^2 - \lambda_i^2)\prod_{j=3}^{5}(\lambda_2^2 - \lambda_j^2)\prod_{p=4}^{5}(\lambda_3^2 - \lambda_p^2)(\lambda_4^2 - \lambda_5^2),$$
$$z_7 = \prod_{i=2}^{6}(\lambda_1^2 - \lambda_i^2)\prod_{j=3}^{6}(\lambda_2^2 - \lambda_j^2)\prod_{p=4}^{6}(\lambda_3^2 - \lambda_p^2)\prod_{q=5}^{6}(\lambda_4^2 - \lambda_q^2)(\lambda_5^2 - \lambda_6^2).$$

Upon simplifying the R.H.S. of above relation, we obtain

$$\sum_{i=1}^{6} r_{1i} = 0. \tag{55}$$

Similarly, we find that

$$\sum_{i=2}^{6} r_{1i}(\lambda_1^2-\lambda_i^2)=0, \sum_{i=3}^{6} r_{1i}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_i^2)\right]=0, \sum_{i=4}^{6} r_{1i}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_i^2)\right]=0, \sum_{i=5}^{6} r_{1i}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_i^2)\right]=0. \tag{56}$$

Also, we have

$$r_{16}\left[\prod_{i=1}^{5}(\lambda_i^2-\lambda_6^2)\right]=1, (\Delta+\lambda_p^2)\varsigma_g(\mathbf{x})=\delta(\mathbf{x})+(\lambda_p^2-\lambda_g^2)\varsigma_g(\mathbf{x}) \;\; p,g=1,.....,6 \tag{57}$$

Now, let us consider

$$\Gamma(\Delta)(\Delta+\lambda_6^2)Y_{11}(\mathbf{x})=\prod_{i=1}^{6}(\Delta+\lambda_i^2)\sum_{g=1}^{6} r_{1g}\varsigma_g(\mathbf{x})=\prod_{i=2}^{6}(\Delta+\lambda_i^2)\sum_{g=1}^{6} r_{1g}[\delta(\mathbf{x})+(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})] \tag{58}$$

Using Eqs. (55)-(57) in the above relation (58), we obtain

$$\begin{aligned}
&\Gamma(\Delta)(\Delta+\lambda_6^2)Y_{11}(\mathbf{x})\\
&=\prod_{i=2}^{6}(\Delta+\lambda_i^2)\sum_{g=2}^{6} r_{1g}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})=\prod_{i=3}^{6}(\Delta+\lambda_i^2)\sum_{g=2}^{6} r_{1g}(\lambda_1^2-\lambda_g^2)\left[\delta(\mathbf{x})+(\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\\
&=\prod_{i=3}^{6}(\Delta+\lambda_i^2)\sum_{g=3}^{6} r_{1g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x})=\prod_{i=4}^{6}(\Delta+\lambda_i^2)\sum_{g=3}^{6} r_{1g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+(\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\\
&=\prod_{i=4}^{6}(\Delta+\lambda_i^2)\sum_{g=4}^{6} r_{1g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x})=\prod_{i=5}^{6}(\Delta+\lambda_i^2)\sum_{g=4}^{6} r_{1g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+(\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\\
&=\prod_{i=5}^{6}(\Delta+\lambda_i^2)\sum_{g=5}^{6} r_{1g}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x})=(\Delta+\lambda_6^2)\sum_{g=5}^{6} r_{1g}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+(\lambda_5^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]=(\Delta+\lambda_6^2)\varsigma_6(\mathbf{x})=\delta(\mathbf{x}).
\end{aligned}$$

Similarly to Eqs. (55)-(57), we obtain

$$\sum_{i=1}^{5} r_{2i}=0, \sum_{i=2}^{5} r_{2i}(\lambda_1^2-\lambda_i^2)=0, \sum_{i=3}^{5} r_{2i}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_i^2)\right]=0, \sum_{i=4}^{5} r_{2i}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_i^2)\right]=0, r_{25}\left[\prod_{i=1}^{4}(\lambda_i^2-\lambda_5^2)\right]=1. \tag{59}$$

Now, we consider the Eq. (54),

$$\begin{aligned}
&\Gamma(\Delta)Y_{44}(\mathbf{x})\\
&=\prod_{i=1}^{5}(\Delta+\lambda_i^2)\sum_{g=1}^{5} r_{2g}\varsigma_g(\mathbf{x})=\prod_{i=2}^{5}(\Delta+\lambda_i^2)\sum_{g=1}^{5} r_{2g}\left[\delta(\mathbf{x})+(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\\
&=\prod_{i=2}^{5}(\Delta+\lambda_i^2)\sum_{g=2}^{5} r_{2g}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})=\prod_{i=3}^{5}(\Delta+\lambda_i^2)\sum_{g=2}^{5} r_{2g}(\lambda_1^2-\lambda_g^2)\left[\delta(\mathbf{x})+(\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\\
&=\prod_{i=3}^{5}(\Delta+\lambda_i^2)\sum_{g=3}^{5} r_{2g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x})=\prod_{i=4}^{5}(\Delta+\lambda_i^2)\sum_{g=3}^{5} r_{2g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+(\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\\
&=\prod_{i=4}^{5}(\Delta+\lambda_i^2)\sum_{g=4}^{5} r_{2g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x})=(\Delta+\lambda_5^2)\sum_{g=4}^{5} r_{2g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+(\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]=(\Delta+\lambda_5^2)\varsigma_5(\mathbf{x})=\delta(\mathbf{x}).
\end{aligned}$$

We introduce the matrix

$$\mathbf{G}(\mathbf{x})=\mathbf{R}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}) \tag{60}$$

From Eqs. (50), (53) and (60), we obtain

$$\mathbf{F}(\mathbf{D_x})\mathbf{G}(\mathbf{x})=\mathbf{F}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x})\mathbf{Y}(\mathbf{x})=\boldsymbol{\Theta}(\Delta)\mathbf{Y}(\mathbf{x})=\delta(\mathbf{x})\mathbf{I}(\mathbf{x})$$

Therefore, $\mathbf{G}(\mathbf{x})$ is a solution to Eq. (32).

Theorem 1: If the condition (31) is satisfied, then the matrix $\mathbf{G}(\mathbf{x})$ defined by the Eq. (60) is the fundamental solution of the system Eqs. (26)-(30) and each element $G_{ij}(\mathbf{x})$ of the matrix $\mathbf{G}(\mathbf{x})$ is represented in the following form:

$$G_{gh}(\mathbf{x}) = R_{gh}(\mathbf{D_x})Y_{11}(\mathbf{x}),\ G_{gq}(\mathbf{x}) = R_{gq}(\mathbf{D_x})Y_{44}(\mathbf{x})\ g = 1,....,7\ h = 1,2,3\ q = 4,....,7 \tag{61}$$

5 BASIC PROPERTIES OF FUNDAMENTAL SOLUTIONS

Theorem 2: Each column of the matrix $\mathbf{G}(\mathbf{x})$ is a solution of system of Eqs. (26)-(30) at every point $\mathbf{x} \in E^3$ except at the origin.

Theorem 3: If the condition (31) is satisfied, then the fundamental solution of the system $\mathbf{F}^*(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$, is the matrix $\mathbf{G}^*(\mathbf{x}) = \left\| G^*_{ij}(\mathbf{x}) \right\|_{7\times 7}$, where

$$\begin{aligned}
&G^*_{gh}(\mathbf{x}) = \frac{1}{\mu}\left[\Delta\,\delta_{gh} - \frac{\lambda'+\mu}{\lambda'+2\mu}\frac{\partial^2}{\partial x_g \partial x_h}\right]Y^*_{11}(\mathbf{x}) = \frac{1}{\lambda'+2\mu}Y^*_{11,gh}(\mathbf{x}) - \frac{1}{\mu}R^*_{gh}Y^*_{11}(\mathbf{x}) = \lambda^*\frac{\delta_{gh}}{|\mathbf{x}|} + \mu^*\frac{x_g x_h}{|\mathbf{x}|^3},\\
&G^*_{44}(\mathbf{x}) = \frac{t_2}{t_1 t_2 - r_1^2}Y^*_{44}(\mathbf{x}), G^*_{45}(\mathbf{x}) = G^*_{54}(\mathbf{x}) = \frac{-r_1}{t_1 t_2 - r_1^2}Y^*_{44}(\mathbf{x}), G^*_{55}(\mathbf{x}) = \frac{t_1}{t_1 t_2 - r_1^2}Y^*_{44}(\mathbf{x}),\\
&G^*_{66}(\mathbf{x}) = \frac{1}{K}Y^*_{44}(\mathbf{x}), G^*_{77}(\mathbf{x}) = \frac{1}{D}Y^*_{44}(\mathbf{x}), G^*_{g;h+3}(\mathbf{x}) = G^*_{g+3;h}(\mathbf{x}) = G^*_{p7}(\mathbf{x}) = G^*_{7p}(\mathbf{x}) = 0,\\
&G^*_{q6}(\mathbf{x}) = G^*_{6q}(\mathbf{x}) = 0, Y^*_{11}(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, Y^*_{44}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \lambda^* = -\frac{\lambda'+3\mu}{8\pi\mu(\lambda'+2\mu)}, \mu^* = -\frac{\lambda'+\mu}{8\pi\mu(\lambda'+2\mu)},\\
&R^*_{gh} = \frac{\partial^2}{\partial x_g \partial x_h} - \Delta\,\delta_{gh}\ \ g,h = 1,2,3\ \ p = 1,....,6\ \ q = 4,5.
\end{aligned} \tag{62}$$

Corollary 1: The relations

$$G^*_{gh}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right),\ G^*_{pq}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right)$$

hold in the neighbourhood of the origin, where $g,h = 1,2,3$ and $p,q = 4,5,6,7.$

Lemma 2: If the condition (31) is satisfied, then

$$\Delta\, w_{p1}(\Delta) = -\frac{1}{\mu}\Gamma(\Delta)\,\delta_{p1} + \frac{1}{M^*}(\Delta + \lambda_6^2)\,N^*_{p1}(\Delta)\ \ p = 1,....,5 \tag{63}$$

We will prove the result for $p = 1$. For $p = 1$,

$$w_{11}(\Delta) = -\frac{1}{M^*\mu}\left[(\lambda'+\mu)\,N^*_{11}(\Delta) - g_1 N^*_{12}(\Delta) - g_2 N^*_{13}(\Delta) + \tau_1 T_0 s_1 N^*_{14}(\Delta) + \tau^1 l_1 N^*_{15}(\Delta)\right]$$

Now

$$\Gamma(\Delta) = \frac{1}{M^*}\det\mathbf{N}(\Delta) = \frac{1}{M^*}\begin{bmatrix}\left((\lambda'+2\mu)\Delta + \rho\omega^2\right)N^*_{11}(\Delta) - g_1\Delta\,N^*_{12}(\Delta) - g_2\Delta\,N^*_{13}(\Delta)\\ +\tau_1 T_0 s_1 \Delta\,N^*_{14}(\Delta) + \tau^1 l_1 \Delta\,N^*_{15}(\Delta)\end{bmatrix}$$

Therefore,

$$\Delta w_{11}(\Delta) = -\frac{1}{M^*\mu}\left[(\lambda' + \mu)\Delta N_{11}^*(\Delta) - g_1 \Delta N_{12}^*(\Delta) - g_2 \Delta N_{13}^*(\Delta) + \tau_1 T_0 s_1 \Delta N_{14}^*(\Delta) + \tau^1 l_1 \Delta N_{15}^*(\Delta)\right]$$
$$= -\frac{1}{M^*\mu}\left[\Gamma(\Delta) M^* - (\mu\Delta + \rho\omega^2) N_{11}^*(\Delta)\right] = -\frac{\Gamma(\Delta)}{\mu} + \frac{(\Delta + \lambda_6^2)}{M^*} N_{11}^*(\Delta)$$

The results for $p = 2,......,5$ can be proved in the similar manner. Let

$$c_{p11} = -\frac{1}{M^*\lambda_p^2} r_{2p} N_{11}^*(-\lambda_p^2), c_{611} = \frac{1}{\rho\omega^2}, c_{pg1} = r_{1p} w_{g1}(-\lambda_p^2), c_{p1g} = r_{2p} w_{1g}(-\lambda_p^2),$$
$$c_{pgh} = r_{2p} w_{gh}(-\lambda_p^2) \;\; p = 1,....,5 \;\; g,h = 2,....,5. \tag{64}$$

Theorem 4: If $\mathbf{x} \neq \mathbf{0}$, then

$$G_{gh}(\mathbf{x}) = \sum_{p=1}^{5} c_{p11}\, \varsigma_{p,gh}(\mathbf{x}) + \mathrm{c}_{611} R_{gh}^* \varsigma_6(\mathbf{x}), \mathrm{G}_{g;e+2}(\mathbf{x}) = \sum_{p=1}^{5} c_{p1e}\, \varsigma_{p,g}(\mathbf{x}),$$
$$\mathrm{G}_{e+2;g}(\mathbf{x}) = \sum_{p=1}^{5} c_{pe1}\, \varsigma_{p,g}(\mathbf{x}), \mathrm{G}_{e+2;y+2}(\mathbf{x}) = \sum_{p=1}^{5} c_{pey}\, \varsigma_p(\mathbf{x}) \;\; \mathrm{g,h} = 1,2,3 \;\; e,y = 2,...,5. \tag{65}$$

From Eq. (57),

$$\Delta \varsigma_p(\mathbf{x}) = -\lambda_p^2\, \varsigma_p(\mathbf{x}) \tag{66}$$

Thus, we have

$$\delta_{gh}\varsigma_p(\mathbf{x}) = -\frac{1}{\lambda_p^2}\left[\Delta\,\delta_{gh}\varsigma_p(\mathbf{x})\right] = -\frac{1}{\lambda_p^2}\left[\frac{\partial^2}{\partial x_g \partial x_h} - R_{gh}^*\right]\varsigma_p(\mathbf{x}) \tag{67}$$

From Eq. (63), we get

$$w_{11}(-\lambda_p^2) - \frac{\Gamma(-\lambda_p^2)}{\mu\,\lambda_p^2} = -\frac{1}{M^*\lambda_p^2}(\lambda_6^2 - \lambda_p^2) N_{11}^*(-\lambda_p^2) \;\; p = 1,......,5 \tag{68}$$

From Eq. (61) with the aid of Eqs. (49), (52) and (66)-(68), we have

$$G_{gh}(\mathbf{x}) = R_{gh}(\mathbf{D_x}) Y_{11}(\mathbf{x}) = \left[\frac{1}{\mu}\Gamma(\Delta)\,\delta_{gh} + w_{11}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}\right]\sum_{p=1}^{6} r_{1p}\varsigma_p(\mathbf{x})$$
$$= \sum_{p=1}^{6} r_{1p}\left[\frac{1}{\mu}\Gamma(-\lambda_p^2)\,\delta_{gh} + w_{11}(-\lambda_p^2)\frac{\partial^2}{\partial x_g \partial x_h}\right]\varsigma_p(\mathbf{x})$$
$$= \sum_{p=1}^{6} r_{1p}\left[-\frac{1}{\mu\lambda_p^2}\Gamma(-\lambda_p^2)\left(\frac{\partial^2}{\partial x_g \partial x_h} - R_{gh}^*\right) + w_{11}(-\lambda_p^2)\frac{\partial^2}{\partial x_g \partial x_h}\right]\varsigma_p(\mathbf{x})$$
$$= \sum_{p=1}^{6} r_{1p}\left[-\frac{1}{M^*\lambda_p^2}(\lambda_6^2 - \lambda_p^2) N_{11}^*(-\lambda_p^2)\frac{\partial^2}{\partial x_g \partial x_h} + \frac{1}{\mu\,\lambda_p^2}\Gamma(-\lambda_p^2) R_{gh}^*\right]\varsigma_p(\mathbf{x}) \tag{69}$$

Now,

$$(\lambda_6^2-\lambda_h^2)\,r_{1h}=r_{2h}\,,\ h=1,......,5\quad\text{and}\quad r_{1p}\,\Gamma(-\lambda_p^2)=\begin{cases}0,\ for\ \ p=1,.....,5\\ 1,\ for\ \ p=6\end{cases}\tag{70}$$

By virtue of Eqs. (64) and (70), Eq. (69) becomes

$$G_{gh}(\mathbf{x})=-\sum_{p=1}^{5}\frac{r_{2p}\,N_{11}^*(-\lambda_p^2)}{M^*\lambda_p^2}\frac{\partial^2}{\partial x_g\,\partial x_h}\varsigma_p(\mathbf{x})+\sum_{p=1}^{6}\frac{1}{\mu\,\lambda_p^2}r_{1p}\Gamma(-\lambda_p^2)\,R_{gh}^*\varsigma_p(\mathbf{x})=\sum_{p=1}^{5}c_{p11}\,\varsigma_{p,gh}(\mathbf{x})+c_{611}R_{gh}^*\varsigma_6(\mathbf{x})$$

Now, consider

$$G_{g;e+2}(\mathbf{x})=R_{g;e+2}(\mathbf{D_x})\,Y_{44}(\mathbf{x})=w_{1e}(\Delta)\frac{\partial}{\partial x_g}\sum_{p=1}^{5}r_{2p}\,\varsigma_p(\mathbf{x})=\sum_{p=1}^{5}r_{2p}\,w_{1e}(-\lambda_p^2)\,\varsigma_{p,g}(\mathbf{x})=\sum_{p=1}^{5}c_{p1e}\,\varsigma_{p,g}(\mathbf{x})$$

and

$$G_{e+2;y+2}(\mathbf{x})=R_{e+2;y+2}(\mathbf{D_x})\,Y_{44}(\mathbf{x})=w_{ey}(\Delta)\sum_{p=1}^{5}r_{2p}\,\varsigma_p(\mathbf{x})=\sum_{p=1}^{5}r_{2p}\,w_{ey}(-\lambda_p^2)\,\varsigma_p(\mathbf{x})=\sum_{p=1}^{5}c_{pey}\,\varsigma_p(\mathbf{x})$$

Similarly the formula for $\mathrm{G}_{e+2;g}(\mathbf{x})$ can be proved.

Lemma 3: If the condition (31) is satisfied, then

$$\sum_{p=1}^{5}r_{2p}=\sum_{p=1}^{5}r_{2p}\,\lambda_p^2=\sum_{p=1}^{5}r_{2p}\,\lambda_p^4=\sum_{p=1}^{5}r_{2p}\,\lambda_p^6=0,\sum_{p=1}^{5}r_{2p}\,\lambda_p^8=1,\sum_{p=1}^{5}\frac{r_{2p}}{\lambda_p^2}=\frac{1}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2\lambda_5^2}=\frac{M^*}{\rho\omega^2N_{11}^*(0)}\tag{71}$$

and

$$\sum_{p=1}^{5}c_{p11}=-\frac{1}{\rho\omega^2},\sum_{p=1}^{5}\lambda_p^2\,c_{p11}=-\frac{1}{\lambda'+2\mu}\tag{72}$$

Using Eq. (52), relations (71) can be proved by direct calculations.

$$N_{11}^*(-\lambda_p^2)=KD\,(t_1t_2-r_1^2)\,\lambda_p^8+M_1^*\lambda_p^6+M_2^*\lambda_p^4+M_3^*\lambda_p^2+N_{11}^*(0),\ p=1,......,5.\tag{73}$$

where M_1^*, M_2^* and M_3^* are coefficients, independent of λ_p and skipped due to lengthy calculations.

From Eqs. (71) and (73), we obtain

$$\sum_{p=1}^{5}\frac{1}{\lambda_p^2}r_{2p}N_{11}^*(-\lambda_p^2)=\sum_{p=1}^{5}r_{2p}\left[KD(t_1t_2-r_1^2)\,\lambda_p^6+M_1^*\lambda_p^4+M_2^*\lambda_p^2+M_3^*+\frac{1}{\lambda_p^2}N_{11}^*(0)\right]=N_{11}^*(0)\sum_{p=1}^{5}\frac{1}{\lambda_p^2}r_{2p}=\frac{M^*}{\rho\omega^2}$$

and

$$\sum_{p=1}^{5}r_{2p}N_{11}^*(-\lambda_p^2)=\sum_{p=1}^{5}r_{2p}\left[KD(t_1t_2-r_1^2)\,\lambda_p^8+M_1^*\lambda_p^6+M_2^*\lambda_p^4+M_3^*\lambda_p^2+N_{11}^*(0)\right]=KD(t_1t_2-r_1^2)$$

Therefore, from (64), we have

$$\sum_{p=1}^{5}c_{p11}=-\frac{1}{M^*}\sum_{p=1}^{5}\frac{1}{\lambda_p^2}r_{2p}N_{11}^*(-\lambda_p^2)=-\frac{1}{\rho\omega^2},\sum_{p=1}^{5}\lambda_p^2c_{p11}=-\frac{1}{M^*}\sum_{p=1}^{5}r_{2p}N_{11}^*(-\lambda_p^2)=-\frac{KD(t_1t_2-r_1^2)}{M^*}=-\frac{1}{(\lambda'+2\mu)}.$$

Theorem 5: The relations

$$G_{ij}(\mathbf{x}) - G_{ij}^{*}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|) \tag{74}$$

hold in neighborhood of the origin, where $i, j = 1,......,7$. Let $\mathbf{x} \neq \mathbf{0}$. From Eqs. (62) and (65), we have

$$\begin{aligned} G_{ij}(\mathbf{x}) - G_{ij}^{*}(\mathbf{x}) &= \frac{\partial^2}{\partial x_i \partial x_j} \sum_{p=1}^{5} c_{p11} \varsigma_p(\mathbf{x}) + c_{611} R_{ij}^{*} \varsigma_6(\mathbf{x}) - \frac{1}{\mu}\left[\Delta \delta_{ij} - \left(1 - \frac{\mu}{\lambda' + 2\mu}\right)\frac{\partial^2}{\partial x_i \partial x_j}\right] Y_{11}^{*}(\mathbf{x}) \\ &= \frac{\partial^2}{\partial x_i \partial x_j}\left[\sum_{p=1}^{5} c_{p11}\varsigma_p(\mathbf{x}) - \frac{1}{\lambda' + 2\mu} Y_{11}^{*}(\mathbf{x})\right] + R_{ij}^{*}\left[\frac{1}{\rho\omega^2}\varsigma_6(\mathbf{x}) + \frac{1}{\mu} Y_{11}^{*}(\mathbf{x})\right] \end{aligned} \tag{75}$$

For $i, j = 1,.2,3$. In the neighborhood of the origin, from Eq. (52), we have

$$\varsigma_p(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}\sum_{g=0}^{\infty}\frac{\left(\iota\lambda_p|\mathbf{x}|\right)^g}{g!} = Y_{44}^{*}(\mathbf{x}) - \frac{\iota\lambda_p}{4\pi} - \lambda_p^2 Y_{11}^{*}(\mathbf{x}) + \bar{Y}_{pp}(\mathbf{x}). \tag{76}$$

where $\bar{Y}_{pp}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}\sum_{g=3}^{\infty}\frac{\left(\iota\lambda_p|\mathbf{x}|\right)^g}{g!}\ \ p = 1,...,6.$ Clearly,

$$\bar{Y}_{pp}(\mathbf{x}) = O\left(|\mathbf{x}|^2\right), \bar{Y}_{pp,j}(\mathbf{x}) = O(|\mathbf{x}|), \bar{Y}_{pp,ij}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|)\ \ i, j = 1,2,3\ \ p = 1,....,6 \tag{77}$$

Consider

$$\sum_{p=1}^{5} c_{p11}\varsigma_p(\mathbf{x}) - \frac{1}{\lambda' + 2\mu} Y_{11}^{*}(\mathbf{x}) = \sum_{p=1}^{5} c_{p11}\left[Y_{44}^{*}(\mathbf{x}) - \frac{\iota\lambda_p}{4\pi} + \bar{Y}_{pp}(\mathbf{x})\right] - \left[\sum_{p=1}^{5}\lambda_p^2 c_{p11} + \frac{1}{(\lambda' + 2\mu)}\right] Y_{11}^{*}(\mathbf{x}) \tag{78}$$

By using equalities (72), from Eq. (78), we have

$$\sum_{p=1}^{5} c_{p11}\varsigma_p(\mathbf{x}) - \frac{1}{\lambda' + 2\mu} Y_{11}^{*}(\mathbf{x}) = -\frac{1}{\rho\omega^2} Y_{44}^{*}(\mathbf{x}) - \frac{\iota}{4\pi}\sum_{p=1}^{5}\lambda_p c_{p11} + \sum_{p=1}^{5} c_{p11}\bar{Y}_{pp}(\mathbf{x}). \tag{79}$$

Also, we have

$$\frac{1}{\rho\omega^2}\varsigma_6(\mathbf{x}) + \frac{1}{\mu} Y_{11}^{*}(\mathbf{x}) = \frac{1}{\rho\omega^2}\left[Y_{44}^{*}(\mathbf{x}) - \frac{\iota\lambda_6}{4\pi} - \lambda_6^2 Y_{11}^{*}(\mathbf{x}) + \bar{Y}_{66}(\mathbf{x})\right] + \frac{1}{\mu} Y_{11}^{*}(\mathbf{x}) = \frac{1}{\rho\omega^2}\left[Y_{44}^{*}(\mathbf{x}) - \frac{\iota\lambda_6}{4\pi} + \bar{Y}_{66}(\mathbf{x})\right] \tag{80}$$

Taking into account (77), (79) and (80) and that $\Delta Y_{44}^{*}(\mathbf{x}) = 0\,(\mathbf{x} \neq 0)$, from Eq. (75), we have

$$\begin{aligned} G_{ij}(\mathbf{x}) - G_{ij}^{*}(\mathbf{x}) &= -\frac{1}{\rho\omega^2}\left[\frac{\partial^2}{\partial x_i \partial x_j} - R_{ij}^{*}\right] Y_{44}^{*}(\mathbf{x}) + \frac{\partial^2}{\partial x_i \partial x_j}\sum_{p=1}^{5} c_{p11}\bar{Y}_{pp}(\mathbf{x}) + \frac{1}{\rho\omega^2} R_{ij}^{*}\bar{Y}_{66}(\mathbf{x}) \\ &= -\frac{1}{\rho\omega^2}\Delta\delta_{ij} Y_{44}^{*}(\mathbf{x}) + \text{constant} + O(|\mathbf{x}|) = \text{constant} + O(|\mathbf{x}|)\ \ i, j = 1,2,3 \end{aligned}$$

Similarly, other formulae of (74) can be proved.

Therefore, matrix $\mathbf{G}^*(\mathbf{x})$ gives the singular part of the fundamental solution $\mathbf{G}(\mathbf{x})$ in the neighborhood of the origin.

6 PARTICULAR CASES

1. If we put $\omega = 0$, that is, taking static case in the Eqs. (26)-(30), we can obtain the fundamental solution of partial differential equations in the generalized theory of thermoelastic diffusion materials with double porosity in case of equilibrium oscillations in terms of elementary functions. In this case, operator $\Gamma(\Delta)$, vector $\hat{\mathbf{\Psi}}(\mathbf{x})$ and the matrix operators $\mathbf{\Theta}(\Delta)$, $\mathbf{R}(\mathbf{D_x})$ and $\mathbf{Y}(\mathbf{x}) = \left\| Y_{ij}(\mathbf{x}) \right\|_{7\times 7}$ are changed in the following forms:

i. $\Gamma(\Delta) = \Delta^3 \prod_{i=1}^{2}(\Delta + \lambda_i^2)$, where $\lambda_i^2, i = 1,2$ are the roots of the equation $\det \mathbf{N}'(-\kappa) = 0$, w.r.t. κ and

$$\mathbf{N}'(\Delta) = \left\| N'_{gh}(\Delta) \right\|_{5\times 5} = \left\| \begin{matrix} \lambda' + 2\mu & -g_1 & -g_2 & 0 & 0 \\ g_1 & t_1\Delta - d_1 & r_1\Delta - \varepsilon_{11} & 0 & 0 \\ g_2 & r_1\Delta - \varepsilon_{11} & t_2\Delta - f_1 & 0 & 0 \\ -s_1 & \xi_{11} & \xi_{22} & K & 0 \\ -l_1 & w & v & 0 & D \end{matrix} \right\|_{5\times 5}$$

ii. $\hat{\mathbf{\Psi}}(\mathbf{x}) = (\mathbf{\Psi}', \Psi_2, \Psi_3, \Psi_4, \Psi_5)$, where

$$\mathbf{\Psi}' = \left[\frac{1}{\mu}\Gamma(\Delta)\mathbf{J} + w_{11}(\Delta)\, grad\, div \right]\mathbf{H} + w_{21}(\Delta)\,\mathrm{grad}\,\mathrm{L} + w_{31}(\Delta)\, grad\, M,$$
$$\Psi_2 = w_{12}(\Delta)\, div\,\mathbf{H} + w_{22}(\Delta) L + w_{32}(\Delta) M,$$
$$\Psi_3 = w_{13}(\Delta)\, div\,\mathbf{H} + w_{23}(\Delta) L + w_{33}(\Delta) M,$$
$$\Psi_4 = w_{14}(\Delta)\, div\,\mathbf{H} + w_{24}(\Delta) L + w_{34}(\Delta) M + w_{44}(\Delta) Z,$$
$$\Psi_5 = w_{15}(\Delta)\, div\,\mathbf{H} + w_{25}(\Delta) L + w_{35}(\Delta) M + w_{55}(\Delta) X,$$

iii. $\mathbf{\Theta}(\Delta) = \left\| \Theta_{pq}(\Delta) \right\|_{7\times 7}$, where

$$\Theta_{ii}(\Delta) = \Gamma(\Delta)\Delta,\ \Theta_{jj}(\Delta) = \Gamma(\Delta), \Theta_{pq}(\Delta) = 0\ \ i = 1,2,3\ \ j = 4,....7\ \ p,q = 1,....,7\ \ p \neq q$$

iv. $\mathbf{R}(\mathbf{D_x}) = \left\| R_{gh}(\mathbf{D_x}) \right\|_{7\times 7}$, where

$$R_{ij}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma(\Delta)\delta_{ij} + w_{11}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j}, R_{i;p+2}(\mathbf{D_x}) = w_{1p}(\Delta)\frac{\partial}{\partial x_i}, R_{e+2;i}(\mathbf{D_x}) = w_{e1}(\Delta)\frac{\partial}{\partial x_i},$$
$$R_{kq}(\mathbf{D_x}) = R_{67}(\mathbf{D_x}) = R_{76}(\mathbf{D_x}) = 0, R_{e+2;p+2}(\mathbf{D_x}) = w_{ep}(\Delta), R_{kk}(\mathbf{D_x}) = w_{k-2;k-2}(\Delta)$$
$$i,j = 1,2,3\ \ p = 2,3,4,5\ \ e = 2,3\ \ k = 6,7\ \ q = 1,....,5.$$

v. $\mathbf{Y}(\mathbf{x}) = \left\| Y_{ij}(\mathbf{x}) \right\|_{7\times 7}$, where

$$Y_{pp}(\mathbf{x}) = r_{11}\varsigma_1^*(\mathbf{x}) + r_{12}\varsigma_2^*(\mathbf{x}) + \sum_{g=1}^{2} r_{1;g+2}\,\varsigma_g(\mathbf{x}), Y_{ee}(\mathbf{x}) = r_{21}\varsigma_1^*(\mathbf{x}) + \sum_{g=1}^{2} r_{2;g+2}\,\varsigma_g(\mathbf{x}),$$
$$Y_{qh}(\mathbf{x}) = 0\ \ p = 1,2,3\ \ e = 4,5,6,7\ \ q,h = 1,........,7,\ q \neq h$$

Here

$$\varsigma_1^*(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \varsigma_2^*(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, r_{11} = -\frac{(\lambda_1^2+\lambda_2^2)(\lambda_1^4+\lambda_2^4)}{\lambda_1^8\lambda_2^8}, r_{12} = r_{21} = \frac{(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)(\lambda_1^3+\lambda_2^3)}{\lambda_1^6\lambda_2^6(\lambda_1+\lambda_2)},$$

$$r_{13} = \frac{1}{\lambda_1^8(\lambda_2^2-\lambda_1^2)}, r_{14} = \frac{1}{\lambda_2^8(\lambda_1^2-\lambda_2^2)}, r_{23} = -\frac{1}{\lambda_1^6(\lambda_2^2-\lambda_1^2)}, r_{24} = -\frac{1}{\lambda_2^6(\lambda_1^2-\lambda_2^2)}.$$

2. If double porosity effect is skipped, then we can construct the fundamental solution of a system of equations in the generalized theory of thermoelastic diffusion as given by Kumar and Kansal [19].
3. If the diffusion effect is neglected, then the fundamental solution of partial differential equations in the generalized theory of thermoelastic materials with double porosity can be obtained similarly given by Scarpetta et al. [17].
4. If further thermal effect is omitted, then we can obtain the fundamental solution of partial differential equations in the theory of elastic materials with double porosity similarly given by Svanadze and de Cicco [15].

7 CONCLUSIONS

The fundamental solution of a system of equations in the generalized theory of thermoelastic diffusion materials with double porosity in case of steady oscillations in terms of elementary functions has been constructed. The fundamental solution makes it possible to investigate three-dimensional boundary value problems of generalized theories of thermoelastic diffusion with double porosity by potential method [20].

REFERENCES

[1] Lord H.W., Shulman Y., 1967, A generalized dynamical theory of thermoelasticity, *Journal of the Mechanics and Physics of Solids* **15**: 299-309.

[2] Nowacki W., 1974, Dynamical problems of thermodiffusion in solids-I, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **22**: 55-64.

[3] Nowacki W., 1974, Dynamical problems of thermodiffusion in solids-II, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **22**: 205-211.

[4] Nowacki W., 1974, Dynamical problems of thermodiffusion in solids-III, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **22**: 257-266.

[5] Nowacki W., 1976, Dynamical problems of diffusion in solids, *Engineering Fracture Mechanics* **8**: 261-266.

[6] Sherief H.H., Hamza F.A., Saleh H.A., 2004, The theory of generalized thermoelastic diffusion, *International Journal of Engineering Science* **42**: 591-608.

[7] Iesan D., 1986, A theory of thermoelastic materials with voids, *Acta Mechanica* **60**: 67-89.

[8] Aouadi M., 2010, A theory of thermoelastic diffusion materials with voids, *Zeitschrift für Angewandte Mathematik und Physik* **61**: 357-379.

[9] Barenblatt G.I., Zheltov I.P., Kochina I.N., 1960, Basic concept in the theory of seepage of homogeneous liquids in fissured rocks (strata), *Journal of Applied Mathematics and Mechanics* **24**: 1286-1303.

[10] Warren J., Root P., 1963, The behavior of naturally fractured reservoirs, *Society of Petroleum Engineers Journal* **3**: 245-255.

[11] Wilson R.K., Aifantis E.C., 1982, On the theory of consolidation with double porosity I, *International Journal of Engineering Science* **20**: 1009-1035.

[12] Iesan D., Quintanilla R., 2014, On a theory of thermoelastic materials with a double porosity structure, *Journal of Thermal Stresses* **37**: 1017-1036.

[13] Kansal T., 2018, Generalized theory of thermoelastic diffusion with double porosity, *Archives of Mechanics* **70**: 241-268.

[14] Svanadze M., 2005, Fundamental solution in the theory of consolidation with double Porosity, *Journal of the Mechanical Behavior of Biomedical Materials* **16**: 123-130.

[15] Svanadze M., De Cicco S., 2013, Fundamental solutions in the full coupled linear theory of elasticity for solid with double porosity, *Archives of Mechanics* **65**: 367-390.

[16] Svanadze M., 2013, Fundamental Solution in the linear theory of consolidation for elastic solids with double porosity, *Journal of Mathematical Sciences* **195**: 258-268.

[17] Scarpetta E., Svanadze M., Zampoli V., 2014, Fundamental solutions in the theory of thermoelasticity for solids with double porosity, *Journal of Thermal Stresses* **37**: 727-748.

[18] Kumar R., Vohra R., Gorla M.G., 2016, Some considerations of fundamental solution in micropolar thermoelastic materials with double porosity, *Archives of Mechanics* **68**: 263-284.

[19] Kumar R., Kansal T., 2012, Plane waves and fundamental solution in the generalized theories of thermoelastic diffusion, *International Journal of Applied Mathematics and Mechanics* **8**: 1-20.

[20] Kupradze V.D., Gegelia T.G., Basheleishvili M.O., Burchuladze T.V., 1979, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Company, Amsterdam, New York, Oxford.